CHAPTER 15: The Principle of Equivalence

15.3 The particle created at \( z_2 \) has mass \( m = h\nu/c^2 \), where \( h \) is Planck’s constant and \( \nu \) is the frequency of the photon. Upon dropping to \( z_1 \) in the gravitational field, the energy is \( mc^2 + mg(z_2 - z_1) \). Thus, the system creates spontaneously an energy \( mg(z_2 - z_1)/c^2 \) in each cycle, unless the photon loses an energy \( h/\nu c^2 \) in moving from \( z_1 \) to \( z_2 \).

15.5 For a difference in height \( h \) between points 1 and 2 the change in time intervals is approximately \( \Delta\tau_1 = \Delta\tau_2 (1 - gh/c^2) \). Taking \( h = 1625 \) m and \( g = 9.8 \) m/s\(^2\) gives

\[
\frac{\Delta\tau_1}{\Delta\tau_2} = 1 - \frac{gh}{c^2} \approx 5.6 \times 10^{-6} \text{ yr}^{-1}.
\]

Thus the two clocks should differ by about 5 \( \mu \)sec per year.

15.8 Apply Kepler’s laws to the orbit, giving

\[
r \simeq 2.7 \times 10^7 \text{ m} \quad \nu \simeq 3.9 \text{ km s}^{-1}.
\]

Defining \( \beta = v/c \), the special relativistic time dilation factor for the satellite is

\[
\gamma = (1 - \beta^2)^{-1/2} \simeq 1 + \frac{1}{2} \beta^2
\]

and we neglect the time dilation for the ground clock produced by Earth’s rotation. The fractional change in frequency is determined by the second term,

\[
\frac{v_s - v_0}{v_0} = -\frac{1}{2}\beta^2 = -8.5 \times 10^{-11},
\]

where the negative sign is because the time is dilated \( (v_s < v_0) \) for the satellite viewed from Earth. For the general relativistic time dilation, Eq. (15.6) gives

\[
\int_{v_0}^{v_s} \frac{d\nu}{\nu} = -\int_{R}^{r_s} \frac{GM}{r^2 c^2} dr
\]

(see also Exercise 15.7). Integrating both sides yields

\[
\frac{v_s}{v_0} = \exp\left[\frac{GM}{c^2} \left( \frac{1}{r_s} - \frac{1}{R} \right) \right] \simeq 1 + \frac{GM}{c^2} \left( \frac{1}{r_s} - \frac{1}{R} \right).
\]

Solving this for the fractional frequency shift gives

\[
\frac{v_s - v_0}{v_0} = \frac{GM}{c^2} \left( \frac{1}{R} - \frac{1}{r_s} \right) = 5.3 \times 10^{-10}.
\]

This is opposite in sign relative to the special relativistic effect and about six times larger. Thus, for every second of elapsed time
• Special relativistic time dilation slows the satellite clock relative to the ground clock by about \(8.5 \times 10^{-11} \times 1 \text{ second} = 0.085 \text{ ns}\).

• Gravitational time dilation (general relativity) slows the ground clock relative to the satellite clock by about \(5.3 \times 10^{-10} \times 1 \text{ second} = 0.53 \text{ ns}\).

The net effect is that for every second the satellite clock gains about \(0.53 - 0.085 = 0.445 \text{ ns}\) relative to the ground clock because of relativistic effects. Suppose that we desire an accuracy of 2 meters from the GPS system for locations on the ground. Light takes 6.7 ns to travel 2 meters. Thus, without the above correction for time dilations, an error in timing that compromises 2-meter resolution will accumulate in about 15 seconds.