Neutron stars are relevant to our discussion of general relativity on two levels.

- They are of considerable intrinsic interest because their quantitative description requires solution of the Einstein equations in the presence of matter.

- In addition, they also explain the existence of pulsars and these in turn provide the most stringent observational tests of general relativity.
19.1 The Oppenheimer–Volkov Equations

Neutron stars have an average density of order $10^{14} - 10^{15}$ g cm$^{-3}$.

- This produces gravitational fields that are of moderate strength by general relativity standards (enormous by Earth standards).
- Escape velocity at the surface is around $\frac{1}{3}c - \frac{1}{2}c$.
- Thus a general relativistic treatment is necessary for their correct description.
- Unlike the vacuum Schwarzschild solution, we must now deal with mass distributions and a finite stress–energy tensor.
- We shall, however, simplify by assuming a static, spherically symmetric configuration for the matter.

**Boundary condition:** With these assumptions we may assume the solution *outside* the neutron star to correspond to the Schwarzschild solution, so the interior solution must match Schwarzschild at the surface.

Thus, we consider the general solution of the Einstein equations for the gravitational field produced by a static, spherical mass distribution that matches the exterior (Schwarzschild) solution at the surface of the spherical mass distribution.
• The matter inside the star is a perfect fluid, with a stress energy
tensor given

\[ T^{\mu}_{\nu} = (\varepsilon + P)u^\mu u_\nu - P\delta^{\mu}_{\nu}, \]

where for later convenience we’ve written tensors in mixed form.

• Spherical symmetry, with a line element of the general form

\[ ds^2 = -e^{\sigma(r)} dt^2 + e^{\lambda(r)} dr^2 + r^2 d\theta^2 + r^2 \sin^2 \theta d\phi^2, \]

implying non-vanishing metric components

\[ g_{00}(r) = -e^{\sigma(r)} \quad g_{11}(r) = e^{\lambda(r)} \quad g_{22}(r) = r^2 \quad g_{33}(r, \theta) = r^2 \sin^2 \theta. \]

Must match smoothly to the Schwarzschild metric at the surface.

• Assumed in equilibrium, so \( \sigma(r) \) and \( \lambda(r) \) are functions only of \( r \) and not of \( t \), and the 4-velocity has no space-like components:

\[ u^\mu = (e^{-\sigma/2}, 0, 0, 0) = (g_{00}^{-1/2}, 0, 0, 0). \]

Inserting these 4-velocity components, the stress–energy tensor takes the diagonal form

\[ T^{\mu}_{\nu} = (\varepsilon + P)u^\mu u_\nu - P\delta^{\mu}_{\nu} = \begin{bmatrix}
-\varepsilon & 0 & 0 & 0 \\
0 & -P & 0 & 0 \\
0 & 0 & -P & 0 \\
0 & 0 & 0 & -P
\end{bmatrix}. \]
• For the vacuum Einstein equation we need only the Ricci tensor to construct the Einstein tensor, but in the general non-vacuum case we need both the Ricci tensor $R_{\mu\nu}$ and the Ricci scalar $R$. It is convenient to express Einstein in mixed tensor form

$$G^\nu_\mu \equiv R^\nu_\mu - \frac{1}{2} \delta^\nu_\mu R = 8\pi T^\nu_\mu.$$ 

Since $T^\nu_\mu$ is diagonal, only diagonal components of $G^\nu_\mu$ needed.

Because of the Bianchi identity

$$G^\mu_{\nu;\mu} = 0$$

and the Einstein equations

$$G^\nu_\mu = 8\pi T^\nu_\mu$$

the stress–energy tensor obeys

$$T^\mu_{\nu;\mu} = 0.$$ 

This implies that we can choose to solve the equation $T^\mu_{\nu;\mu} = 0$ in place of solving one of the Einstein equations. In many cases this can lead to a faster solution than solving all the Einstein equations directly.

We shall employ that strategy here, using two Einstein equations and the constraint equation $T^\mu_{\nu;\mu} = 0$ in mixed-tensor form to obtain a solution.
The constraint equation has been solved in Exercise 17.19, where you were asked to show that

$$T^\mu_{\nu;\mu} = 0 \quad \rightarrow \quad P' + \frac{1}{2}(P + \rho)\sigma' = 0.$$  

(primes denoting partial derivatives with respect to $r$) for a metric and stress–energy tensor

$$ds^2 = -e^{\sigma(r)}dt^2 + e^{\lambda(r)}dr^2 + r^2 d\theta^2 + r^2 \sin^2 \theta d\phi^2,$$

$$T^\mu_{\nu} = \begin{bmatrix} -\varepsilon & 0 & 0 & 0 \\ 0 & -P & 0 & 0 \\ 0 & 0 & -P & 0 \\ 0 & 0 & 0 & -P \end{bmatrix}$$

We require two additional equations, with the simplest choices being

$$G^0_0 = 8\pi T^0_0 \quad G^1_1 = 8\pi T^1_1$$

The Einstein tensors $G_{00}$ and $G_{11}$ were derived for this metric in Exercise 17.1 and tabulated in Appendix C. Using contraction with the metric tensor to raise an index we obtain from those results

$$G^0_0 = g^{00}G_{00} = -e^{-\sigma}G_{00} = e^{-\lambda} \left( \frac{1}{r^2} - \frac{\lambda'}{r} \right) - \frac{1}{r^2}$$

$$G^1_1 = g^{11}G_{11} = e^{-\lambda}G_{11} = e^{-\lambda} \left( \frac{1}{r^2} + \frac{\sigma'}{r} \right) - \frac{1}{r^2}.$$
From the preceding equations we find then that we must solve

\[ G^0_0 = - e^{-\lambda} \left( \frac{1}{r^2} - \frac{\lambda'}{r} \right) + \frac{1}{r^2} = 8\pi \varepsilon(r) \]

\[ G^1_1 = e^{-\lambda} \left( \frac{1}{r^2} - \frac{\sigma'}{r} \right) - \frac{1}{r^2} = 8\pi P(r) \]

\[ P' + \frac{1}{2} (P + \rho) \sigma' = 0. \]

To proceed we note that the first Einstein equation may be rewritten as

\[ G^0_0 = \frac{1}{r^2} \frac{d}{dr} \left[ r \left( 1 - e^{-\lambda} \right) \right] = \frac{2}{r^2} \frac{dm}{dr} = 8\pi \varepsilon, \]

where we have defined a new parameter

\[ 2m(r) \equiv r(1 - e^{-\lambda}). \]

At this point \( m(r) \) is only a reparameterization of the metric coefficient \( e^{\lambda} \) since, upon multiplying by \( e^{\lambda} \),

\[ e^{\lambda} \left( \frac{r}{r - 2m(r)} \right) = \left( 1 - \frac{2m(r)}{r} \right)^{-1}, \]

but \( m(r) \) will be interpreted below as the total mass–energy enclosed within the radius \( r \). With this interpretation we note that \( e^{\lambda} \) is of the Schwarzschild form for \( r \) outside the spherical mass distribution of the star.
from the first Einstein equation

\[ G^0_0 = \frac{2}{r^2} \frac{dm}{dr} = 8\pi \varepsilon \quad \rightarrow \quad dm = 4\pi r^2 \varepsilon dr, \]

and thus

\[ m(r) = 4\pi \int_0^r \varepsilon(r) r^2 \, dr, \]

with an integration constant \( m(0) = 0 \) chosen on physical grounds.

Now consider the second Einstein equation

\[ e^{-\lambda} \left( \frac{1}{r^2} - \frac{\sigma'}{r} \right) - \frac{1}{r^2} = 8\pi P(r) \]

Solving it for \( \sigma' = d\sigma/dr \) gives

\[ \frac{d\sigma}{dr} = e^\lambda \left( 8\pi r P(r) + \frac{1}{r} \right) - \frac{1}{r}, \]

and substitution of

\[ e^\lambda = \frac{r}{r - 2m(r)} \]

leads to

\[ \frac{d\sigma}{dr} = \frac{8\pi r^3 P(r) + 2m(r)}{r(r - 2m(r))}. \]

Therefore the preceding two equations may be used to define the metric coefficients \( e^\sigma \) and \( e^\lambda \) in terms of the parameter \( m(r) \) and the pressure \( P(r) \).
Finally, we may combine
\[ P' + \frac{1}{2}(P + \rho)\sigma' = 0 \quad \text{and} \quad \frac{d\sigma}{dr} = \frac{8\pi r^3 P(r) + 2m(r)}{r(r - 2m(r))} \]
to give
\[ \frac{dP}{dr} = -\frac{(P(r) + \varepsilon(r))(4\pi r^3 P(r) + m(r))}{r(r - 2m(r))}. \]

Collecting our results, we have obtained the Oppenheimer–Volkov equations for the structure of a static, spherical, gravitating perfect fluid
\[ \frac{dP}{dr} = \frac{(P(r) + \varepsilon(r))(m(r) + 4\pi r^3 P(r))}{r^2 \left(1 - \frac{2m(r)}{r}\right)}, \]
\[ m(r) = 4\pi \int_0^r \varepsilon(r) r^2 \, dr \]
where \( m(r) \) is the total mass contained within a radius \( r \).
\[ \frac{dP}{dr} = \frac{(P(r) + \varepsilon(r))(m(r) + 4\pi r^3 P(r))}{r^2 \left(1 - \frac{2m(r)}{r}\right)}, \]

\[ m(r) = 4\pi \int_0^r \varepsilon(r) r^2 dr \]

- Solution of these equations requires specification of an *equation of state* that relates the density to the pressure.
- They may then be integrated from the origin outward with initial conditions \( m(r = 0) = 0 \) and an arbitrary choice for the central density \( \varepsilon(r = 0) \) until the pressure \( P(r) \) becomes zero.
- This defines the surface of the star \( r = R \), with the mass of the star given by \( m(R) \).
- For a given equation of state each choice of \( \varepsilon(0) \) will give a unique \( R \) and \( m(R) \) when the equations are integrated.
- This defines a *family of solutions* characterized by
  - a specific equation of state and
  - the value of a single parameter (the central density, or a quantity related to it like central pressure).
These equations represent the general relativistic (covariant) description of hydrostatic equilibrium for a spherical, gravitating perfect fluid.

- The condition of hydrostatic equilibrium was built into the solution through the assumption

\[ u^\mu = (e^{-\sigma/2}, 0, 0, 0) = \left(g_0^{-1/2}, 0, 0, 0\right). \]

which constrains the fluid to be static since the 4-velocity has no non-zero space components.

- They reduce to the Newtonian description of hydrostatic equilibrium in the limit of weak gravitational fields

However, the Oppenheimer–Volkov equations imply significant deviations from the Newtonian description in strong gravitational fields such as those for neutron stars. To see this clearly, we may rewrite them in the form (Exercise)

\[
4\pi r^2 dP(r) = \frac{-m(r)dm(r)}{r^2}
\]

\[
\times \left(1 + \frac{P(r)}{\varepsilon(r)}\right) \left(1 + \frac{4\pi r^3 P(r)}{m(r)}\right) \left(1 - \frac{2m(r)}{r}\right)^{-1}
\]

\[ dm(r) = 4\pi r^2 \varepsilon(r) dr. \]

These equations may be interpreted in the following way:
19.1. THE OPPENHEIMER–VOLKOV EQUATIONS

\[
\frac{4\pi r^2 dP(r)}{\text{Force acting on shell}} = \frac{-m(r)dm(r)}{r^2} \text{ Newtonian}
\]

\[
\times \left(1 + \frac{P(r)}{\varepsilon(r)} \frac{\text{GR}}{\text{GR}}\right) \left(1 + \frac{4\pi r^3 P(r)}{m(r)} \frac{\text{GR}}{\text{GR}}\right) \left(1 - \frac{2m(r)}{r} \frac{\text{GR}}{\text{GR}}\right)^{-1}
\]

\[
dM(r) = \frac{4\pi r^2 \varepsilon(r) dr}{\text{Mass–energy of shell}}.
\]

- The second equation gives the mass–energy of a shell lying between radii \(r\) and \(r + dr\).
- The left side of the first equation is the net force acting outward on this shell.
- The first factor on the right side of the first equation is the attractive Newtonian gravity acting on the shell because of the mass interior to it.
CHAPTER 19. NEUTRON STARS AND GENERAL RELATIVITY

\[ 4\pi r^2 dP(r) = \frac{-m(r) dm(r)}{r^2} \]

Force acting on shell

\[ \times \left( 1 + \frac{P(r)}{\varepsilon(r)} \right) \left( 1 + \frac{4\pi r^3 P(r)}{m(r)} \right) \left( 1 - \frac{2m(r)}{r} \right)^{-1} \]

\[ dm(r) = 4\pi r^2 \varepsilon(r) dr \]

• The last three factors on the right side of the first equation—the factors on the second line—represent general relativity effects causing deviation from Newtonian gravitation.

• Since all three factors on the second line of the first equation exceed unity as the star becomes relativistic, we find that gravity in general relativity is consistently stronger than in the corresponding Newtonian description of the same problem.

Gravity is enhanced by coupling to pressure in the general relativistic description. This will ultimately imply that there are fundamental limiting masses for strongly gravitating objects.
19.2 Interpretation of the Mass Parameter

The parameter $m(r)$ entering the Oppenheimer–Volkov equations has been interpreted provisionally as the total mass–energy enclosed within a radius $r$. Let us now provide some more substantial justification for this interpretation.

1. Outside a star of radius $R$, $m(r)$ becomes equal to $m(R)$, which is the mass that would be detected through Kepler’s law for the orbital motion if the star were a component of a well-separated binary system.

2. In the Newtonian limit, it is clear from

$$m(r) = 4\pi \int_0^r \varepsilon(r)r^2 \, dr,$$

that $m(r)$ can be unambiguously interpreted as the mass contained within the radius $r$.

3. For relativistic stars $m(r)$ may be consistently split into a contribution from a rest mass $m_0(r)$, an internal energy $U(r)$, and a gravitational energy $\Omega(r)$,

$$m(r) = m_0(r) + U(r) + \Omega(r),$$

as we now demonstrate. Formally we can split the energy density $\varepsilon$ into a contribution from the rest mass and one from internal energy,

$$\varepsilon = \mu_0 n + (\varepsilon - \mu_0 n),$$

where the first term is the total rest mass of $n$ particles of average mass $\mu_0$ and the second term in parentheses is the contribution of internal energy. The proper volume for a spherical shell of thickness $dr$ is

$$dV = 4\pi r^2 \sqrt{g_{11}} \, dr = 4\pi r^2 \sqrt{e^\lambda} \, dr = 4\pi r^2 (1 - 2m/r)^{-1/2} \, dr.$$
Thus the total rest mass inside the radius $r$ is

$$m_0(r) = \int_0^r \mu_0 n \, dV = 4\pi \int_0^r r^2 (1 - 2m/r)^{-1/2} \mu_0 n \, dr,$$

the total internal energy inside $r$ is

$$U(r) = \int_0^r (\rho - \mu_0 n) \, dV = 4\pi \int_0^r r^2 (1 - 2m/r)^{-1/2} (\rho - \mu_0 n) \, dr,$$

and the total mass–energy inside $r$ is

$$m(r) = 4\pi \int_0^r \epsilon(r) r^2 \, dr.$$

Thus, the difference

$$\Omega(r) = m(r) - m_0(r) - U(r)$$

$$= -4\pi \int_0^r r^2 \rho \left(1 - (1 - 2m/r)^{-1/2}\right) \, dr$$

must be the total gravitational energy inside $r$.

These observations give us some confidence that $m(r)$ may indeed be interpreted as the total mass–energy inside the coordinate $r$. 
19.3 Some Quantitative Estimates for Neutron Stars

Detailed properties of neutron stars require numerical solution of the Oppenheimer–Volkov equations with realistic equations of state. However, many of their basic properties can be estimated by employing these equations, or even Newtonian concepts, in simpler ways (see Exercises).

- The simple assumption that in a neutron star gravity packs the neutrons down to their hard-core radius of order $10^{-13}$ cm yields that
  - The most massive neutron stars contain about $3 \times 10^{57}$ baryons (mostly neutrons) within a radius of about 7 km, with a mass of about $2.3 M_\odot$.
  - This implies an average density $> 10^{15}$ g cm$^{-3}$ (several times nuclear matter density).
  - This implies a (gravitational) binding energy $\sim 100$ MeV (order of magnitude larger than the binding energy of nucleons in nuclear matter).

- The total gravitational binding energy is within an order of magnitude of the rest mass energy, and the escape velocity is $\sim 50\%$ of the speed of light. Both indicate that general relativistic effects are significant.
Although general relativity is important for the overall properties of neutron stars, over a microscopic scale characteristic of nuclear and other sub-atomic interactions the metric is essentially constant.

- Thus the microphysics (nuclear and elementary particle interactions) of the neutron star can be described by quantum mechanics implemented in flat space-time (special relativistic quantum field theory).

- For neutron stars it is possible to decouple gravity (which governs the overall structure) from quantum mechanics (which governs the microscopic properties).
19.4 The Binary Pulsar

The Binary Pulsar PSR 1913+16 (Hulse–Taylor pulsar) was discovered using the Arecibo 305 meter radio antenna.

- It is about 5 kpc away, near the boundary of the constellations Aquila and Sagitta.

- This pulsar rotates 17 times a second, giving a pulsation period of 59 milliseconds.

- It is in a binary system with another neutron star (not a pulsar), with a 7.75 hour period.

- The precise repetition frequency of the pulsar means that it is basically a very high quality clock orbiting in a binary system that feels very strong, time-varying gravitational effects.

→ Precise tests of general relativity
19.4.1 Periodic Variations

The repetition period for a pulsar is associated with the spin of the pulsar and is atomic clock-like in its precision. Thus

- Variations in that period as observed from Earth must be associated with orbital motion in the binary.

- These variations can be used to give very precise information about the orbit.

- When the pulsar is moving toward us, the repetition rate of the pulses as observed from Earth will be higher than when the pulsar is moving away (Doppler effect), and this can be used to measure the radial velocity (see Fig. 19.1).
The pulse arrival times vary as the pulsar moves through its orbit

- It takes three seconds longer for the pulses to arrive from the far side of the orbit than from the near side.

- From this, the Binary Pulsar orbit can be inferred to be about a million kilometers (three light seconds) further away from Earth when on the far side of its orbit than when on the near side.
19.4.2 Orbital Characteristics

The orbits determined for the binary system are shown in Fig. 19.2.

- Each neutron star has a mass of about 1.4 solar masses.
- The orbits are very eccentric (eccentricity \( \sim 0.6 \)).
- The minimum separation (periastron) is about 1.1 solar radii.
- The maximum separation (apastron) is about 4.8 solar radii.
- The orbital plane is inclined by about 45 degrees, as viewed from Earth.
• By Kepler’s laws, the radial velocity of the pulsar varies substantially as it moves around its elliptical orbit, as illustrated in Fig. 19.1 earlier.

• These orbits are not quite closed ellipses because of precession effects associated with general relativity.
  
  – This causes the location of the periastron to shift a small amount for each revolution (Fig. 19.3).
  
  – The points P1, P2, and P3 are periastrons on three successive orbits (with the amount of precession greatly exaggerated for clarity).
19.5 Precision Tests of General Relativity

The discovery and study of the Binary Pulsar was of such fundamental importance that Taylor and Hulse were awarded the Nobel Prize in Physics for their work (only Nobel ever given for relativity). Chief among the reasons for this importance is that the Binary Pulsar has provided the most stringent tests of general relativity available before the discovery of the Double Pulsar.
19.5.1 Precession of Orbits

Because spacetime is warped by the gravitational field in the vicinity of the pulsar, the orbit will precess with time.

- This is the same effect as the precession of the perihelion of Mercury, but it is much larger for the present case.
- The Binary Pulsar’s periastron advances by 4.2 degrees per year, in accord with the predictions of general relativity.
- In a single day the orbit of the Binary Pulsar advances by as much as the orbit of Mercury advances in a century!

19.5.2 Time Dilation

- When the binary pulsar is near periastron, gravity is stronger and its velocity is higher and time should run slower.
- Conversely, near apastron the field is weaker and the velocity lower, so time should run faster.
- It does both, in the amount predicted by GR.
19.5.3 Emission of Gravitational Waves

The revolving pair of masses is predicted by general relativity to radiate gravitational waves, causing the orbit to shrink (Fig. 19.4).

- The time of periastron can be measured very precisely and is found to be shifting.
- This shift corresponds to a decrease in the orbital period by 76 millionths of a second per year.
- The corresponding decrease in the size of the orbit by about 3.3 millimeters per revolution.
The quantitative decrease in periastron time is illustrated by the data points in the above figure.

- Because the orbital period is short, the shift in periastron arrival time has accumulated to more than 30 seconds (earlier) since discovery.

- This decay of the size of the orbit is in agreement with the amount of energy that general relativity predicts should be leaving the system in the form of gravitational waves (dashed line in figure).

- Although gravitational waves have not yet been detected directly, precision measurements on the Binary Pulsar give strong indirect evidence for the correctness of this key prediction of general relativity.
19.6 Origin and Fate of the Binary Pulsar

Formation of a neutron star binary is not easy. One of two things must happen

- A binary must form with two stars massive enough to become supernovae and produce neutron stars, and the neutron stars thus formed must remain bound to each other through the two supernova explosions.

- The neutron star binary must result from gravitational capture of one neutron star by another.

These are improbable events, but not impossible, and the existence of the Binary Pulsar (and several similar systems) demonstrates empirically that mechanisms exist for it to happen.
Once a neutron star binary is formed its orbital motion radiates energy as gravitational waves, the orbits must shrink, and eventually the two neutron stars must merge.

- Because of the gravitational wave radiation and the corresponding shrinkage of the Binary Pulsar orbit (3.3 millimeters per revolution), merger is predicted in about 300 million years.

- The sum of the masses of the two neutron stars is likely above the critical mass to form a black hole. Therefore, the probable fate of the Binary Pulsar is merger and collapse to a rotating (Kerr) black hole.

- As two neutron stars in a binary approach each other they will revolve faster (Kepler’s third law).

- This will cause them to emit gravitational radiation more rapidly, which will in turn cause the orbit to shrink even faster.

- Thus, near the end the merger of two neutron stars will proceed rapidly in a positive-feedback runaway and will emit very strong gravitational waves that may be detectable with current-generation gravitational wave detectors.

These considerations are valid for any binary star system, not just the Binary Pulsar, but the gravitational wave effects are more pronounced for binaries involving highly compact objects like neutron stars.
19.7 The Double Pulsar

In 2003 a binary neutron star system (the Double Pulsar) was discovered in which both neutron stars were observed as pulsars in a very tight, partially eclipsing orbit (Fig. 19.5).

- The two neutron stars have masses of $1.3381 \pm 0.0007 M_\odot$ (component A), and $1.2489 \pm 0.0007 M_\odot$ (component B).
- They have spin periods of 22.7 ms (component A) and 2.77 s (component B).
- The orbit is slightly eccentric ($\varepsilon = 0.088$).
- The orbit has a mean radius of about 800,000 km ($1.25 R_\odot$).
- Thus the orbital period is only 147 minutes, with a mean orbital velocity of about $10^6 \text{ km h}^{-1}$.
- The very fast orbital period and the exquisite timing associated with the pulsar clocks has allowed the Double Pulsar to give the most precise tests of general relativity to date.