

Exercise Solutions

8.2 For a mass element \( m \) near the surface of the star, requiring that the centrifugal force be much less that the gravitational force acting on it gives

\[
m \omega^2 R \ll \frac{GMm}{R^2} \quad \rightarrow \quad \omega \ll \sqrt{\frac{GM}{R^3}},
\]

where \( \omega \) is the rotational velocity. But the hydrodynamical timescale is

\[
\tau_{\text{hydro}} \simeq \sqrt{\frac{1}{G \bar{\rho}}} \simeq \sqrt{\frac{R^3}{GM}}.
\]

Thus rotational flattening should be negligible if \( \omega \ll (1/\tau_{\text{hydro}}) \). For the Sun \( \tau_{\text{hydro}} \sim 1 \) hour and the rotational period is about a month, which gives \( \omega/(1/\tau_{\text{hydro}}) \sim 1.5 \times 10^{-3} \). On the other hand, for Saturn the rotational period is 614 minutes and the hydrodynamical timescale is about 78 minutes if we view it as a gas in hydrodynamical equilibrium with an average density of 0.7 g cm\(^{-3}\). Then \( \omega/(1/\tau_{\text{hydro}}) \sim 0.12 \). Thus Saturn should be much more susceptible to rotational flattening than the Sun, as observed.

8.6 In \( \hbar = c = 1 \) units the energy is given by \( E^2 = p^2 + m^2 \). Thus,

\[
E_1 - E_2 = \frac{(E_1 + E_2)(E_1 - E_2)}{E_1 + E_2} = \frac{E_1^2 - E_2^2}{E_1 + E_2} = \frac{p_1^2 + m_1^2 - p_2^2 - m_2^2}{E_1 + E_2}
\]

\[
\simeq \frac{m_1^2 - m_2^2}{E_1 + E_2} \simeq \frac{m_1^2 - m_2^2}{2E} = \frac{\Delta m^2}{2E},
\]

where \( E_1 \sim E_2 \equiv E \), we have defined \( \Delta m^2 \equiv m_2^2 - m_1^2 \), and where we have assumed coherent states with \( p_1 \sim p_2 \). The distance traveled in a time \( t \) is \( r \simeq t \), and is essentially the same for \( \nu_1 \) and \( \nu_2 \) if the state is coherent. Therefore, from these results and the trigonometric identity \( 1 - \cos 2\alpha = 2\sin^2 \alpha \),

\[
1 - \cos(E_2 - E_1)t = 1 - \cos\left(\frac{\Delta m^2}{2E}t\right) = 2\sin^2\left(\frac{\Delta m^2}{4E}t\right) = 2\sin^2\left(\frac{\pi r}{L}\right),
\]

where we have defined the oscillation length \( L \equiv 4\pi E/\Delta m^2 \). This permits the probabilities (8.32) to be rewritten in terms of the oscillation length \( L \) and the distance \( r \) traveled in a time \( t \), which gives Eq. (8.36).

8.7 In a 2-flavor model the electron and muon neutrino flavor eigenstates are

\[
|\nu_e \rangle = |\nu_1 \rangle \cos \theta + |\nu_2 \rangle \sin \theta \quad |\nu_\mu \rangle = -|\nu_1 \rangle \sin \theta + |\nu_2 \rangle \cos \theta
\]

in terms of the mass eigenstates \( |\nu_1 \rangle \) and \( |\nu_2 \rangle \). The pure \( \nu_e \) state evolves after a time \( t \) to

\[
|\nu(t)\rangle = |\nu_1(0)\rangle \cos \theta e^{-iE_1t} + |\nu_2(0)\rangle \sin \theta e^{-iE_2t}.
\]
Use the first two equations to solve for the mass eigenstates,

\[ |\nu_1 \rangle = \cos \theta |\nu_e \rangle - \sin \theta |\nu_\mu \rangle \quad |\nu_2 \rangle = \sin \theta |\nu_e \rangle + \cos \theta |\nu_\mu \rangle , \]

and insert this into the equation for \( |\nu(t)\rangle \) to give

\[ |\nu(t)\rangle = (\cos^2 \theta e^{-iE_1 t} + \sin^2 \theta e^{-iE_2 t}) |\nu_e \rangle + \sin \theta \cos \theta (e^{-iE_2 t} + e^{-iE_1 t}) |\nu_\mu \rangle. \]

The probability after a time \( t \) to observe a \( \nu_e \) if we started with a \( \nu_e \) is then

\[
P(\nu_e \rightarrow \nu_e, t) = |\langle \nu_e | \nu(t) \rangle|^2 = \cos^4 \theta + \sin^4 \theta + \sin^2 \theta \cos^2 \theta (e^{i(E_2 - E_1) t} + e^{-i(E_2 - E_1) t}),
\]

where \( \langle \nu_e | \nu_e \rangle = 1 \) and \( \langle \nu_e | \nu_\mu \rangle = 0 \) have been employed. Using the identities

\[ e^{ix} + e^{-ix} = 2 \cos x \quad \cos^4 \theta + \sin^4 \theta = 1 - 2 \sin^2 \theta \cos^2 \theta \quad \sin \theta \cos \theta = \frac{1}{2} \sin(2\theta), \]

this finally may be simplified to

\[
P(\nu_e \rightarrow \nu_e, t) = 1 - \frac{1}{2} \sin^2 2\theta (1 - \cos([(E_2 - E_1) t]).
\]

By analogous steps we obtain for the probability of observing a \( \nu_\mu \) if we started with a \( \nu_e \),

\[
P(\nu_e \rightarrow \nu_\mu, t) = |\langle \nu_\mu | \nu(t) \rangle|^2 = \frac{1}{2} \sin^2 2\theta (1 - \cos([(E_2 - E_1) t]),
\]

which also could have been obtained immediately by the condition

\[
P(\nu_e \rightarrow \nu_\mu, t) = 1 - P(\nu_e \rightarrow \nu_e, t)
\]

implied by unitarity in the 2-flavor space.