One commonplace of modern astronomy that would have been highly perplexing for ancient astronomers is that many stars vary their light output by detectable amounts over time.

- In some cases these variations are asynchronous and in others they are highly periodic.
- They may be so small as to require precise instruments to detect them, or sufficiently large that they are easily visible to the naked eye.
These *variable stars* may be loosely classified into three categories.

1. *Eclipsing binaries* are binary stars in which the total light output of the system is altered by geometrical eclipses of one star by the other. If the binary system is too far away to resolve the two components, this will appear to be a single star with periodic variation in light output.

2. *Eruptive and exploding variables* are stars that suddenly increase light output and often eject mass because of a rapid and violent disruption or partial disruption of the star. Novae and supernovae are dramatic examples in this category.

3. *Pulsating variable stars* appear to undergo (possibly complex) pulsations that alter the light output in periodic or irregular fashion, without disrupting significantly the overall structure of the star. Well-known examples of this category are Cepheid variables and RR-Lyrae stars.

In this chapter we examine in more depth this latter category and the reasons that some stars become unstable against pulsations for certain periods of their lives.
Some common classes of pulsating variable stars and their characteristics are given in Table 10.1.
Variable stars are found in specific regions of the Hertzsprung–Russell diagram, as illustrated in Fig. 10.1.

- There we see that many types of pulsating variables are confined to a narrow, rather vertical, strip in the HR diagram called the instability strip.

- This suggests that there is a fundamental mechanism operating in a variety of stars
  
  - in different luminosity classes, but
  
  - over a relatively narrow surface temperature range,

  that leads to pulsational instability.
10.2 Adiabatic Radial Pulsations

At the simplest level we may examine stellar pulsation in terms of oscillations within the body of the star that are

- Adiabatic
- Linear in the displacement
- Maintain spherical symmetry for the star.

Such an analysis has much in common with the study of small-amplitude vibrations in other physical systems:

- The pulsations are treated as as free radial vibrations.
- Gas compression plays the role of a spring constant.
- One finds that stars disturbed slightly from spherical hydrostatic equilibrium exhibit discrete vibrational frequencies that are called radial acoustic modes.
10.2.1 Radial Acoustic Modes

It usually proves convenient in discussions of stellar pulsation to work in Lagrangian coordinates, where $m(r)$ is the independent variable and corresponds to the mass contained within a radius $r$.

- Then if we expand the pressure, radial coordinate, and density as time-dependent oscillations around the equilibrium values (which are denoted by a subscript zero),
  \[
P(m,t) = P_0(m) \left( 1 + \delta P(m) e^{i\omega t} \right)
  \]
  \[
r(m,t) = r_0(m) \left( 1 + \delta r(m) e^{i\omega t} \right)
  \]
  \[
\rho(m,t) = \rho_0(m) \left( 1 + \delta \rho(m) e^{i\omega t} \right),
\]
  the radial displacement $\delta r(m)$ is described by the differential equation
  \[
\frac{d^2(\delta r)}{dr_0^2} + \left( \frac{4}{r_0} - \frac{\rho_0 g_0}{P_0} \right) \frac{d(\delta r)}{dr_0} + \frac{\rho_0}{\Gamma_1 P_0} \left[ \omega^2 + (4 - 3\Gamma_1) \frac{g_0}{r_0} \right] \delta r = 0,
\]
  where $\Gamma_1$ is an adiabatic exponent, $g_0 \equiv Gm/r_0^2$, and $\omega$ is the adiabatic oscillation frequency.

- This equation must be solved with two boundary conditions, one at the center of the star and one at the surface.
  - At the center one requires $d(\delta r)/dr_0 = 0$.
  - The simplest physically reasonable surface boundary condition is to require $\delta PP_0 = 0$, though more complicated ones can be used.

Exercises 10.3 and 11.5 give examples of applying this equation.
Most intrinsically variable stars are pulsing in radial acoustic modes, which correspond to standing waves within the star.

- The *fundamental mode* has no nodes (points of zero motion) between the center and surface, implying that the stellar matter involved in the vibration all moves in the same direction at a given time.

- The *first overtone* has one node between the center and the surface, so the matter moves in one direction outside this node and in the opposite direction inside this node at a given phase of the pulsation.

- Likewise, higher overtones with additional nodes and more complex motion may be defined.

- Just as for musical instruments and other acoustically vibrating systems, a star may exhibit several modes of oscillation at once.

- The physical motion of the gas in radial stellar pulsations is largest in the fundamental mode and is considerably smaller in the first overtone.

- In higher overtones the motion of the gas in an oscillation cycle is even smaller.

Pulsating variables appear to be oscillating primarily in the *fundamental mode and/or the first overtone.*
• It is thought that most Classical and Type II Cepheids oscillate in the fundamental mode,

• RR Lyrae stars oscillate in either the fundamental mode or first overtone (or both).

• For long-period red variables the evidence is less conclusive and they may pulsate in either the fundamental or first overtone modes.
Pulsations in realistic stars are more complicated than the linear adiabatic analysis discussed in the preceding paragraphs would indicate.

- For example both the
  1. rate of energy production
  2. rate of internal energy transport

may be modified by pulsations, so we may expect that they are not completely adiabatic and must examine deviations from adiabaticity.

- In particular, we must ask the question:

  What energy input sustains the pulsation modes of a pulsating variable star?
Eddington first examined systematically the idea that stellar pulsations are free radial oscillations, but realized that dissipation processes in the gas would damp out such oscillations quickly.

- Example: pulsations of Cepheid variables should be damped on a timescale of order $10^4$ years without some mechanism to amplify and sustain the oscillations.

- Thus the steady, long-term pulsing of a Cepheid variable
  - cannot be due to a one-time excitation of eigen-modes and
  - cannot be adiabatic.

Eddington proposed that pulsating variable stars are a form of *heat engine* continuously transforming thermal energy into the mechanical energy of the pulsation.
On the other hand, it will turn out that in realistic stars the pulsation may often be approximated as *nearly adiabatic*:

- Instabilities grow on a timescale that is *long relative to the time for one pulsation.*
- Over one acoustic oscillation cycle (which is essentially a *hydrodynamic timescale*), the amount of heat exchanged is typically small.
- This is because energy transfer occurs on a *Kelvin–Helmholtz timescale*, which is much longer than the hydrodynamic timescale.
- Therefore after a single acoustic cycle the star returns almost—*but not quite*—to the initial state.
- The “*not quite*” measures the lack of reversibility and therefore the non-adiabaticity of the process.

With this as introduction, let us now consider the role of non-adiabatic effects in sustaining stellar pulsation.
10.3 Non-Adiabatic Radial Pulsations

For each layer of the star a net amount of work is done during a pulsation cycle that must be equal to the difference of the heat flowing into that layer and that flowing out.

- If the oscillation is to be self-sustaining for a single layer, we must have a mechanism whereby heat
  - enters the layer at *high temperature* and
  - leaves at *low temperature*.

- If layers driving the pulsation *absorb energy near the time of maximum compression*, the oscillations will be amplified because the time of maximum pressure in the layer will occur after maximum compression.

  This is similar to the reason that it is optimal to fire the spark plug near the *end* of the compression stroke in an internal combustion engine.

- A sustained oscillation for a significant part of the star then requires that a set of different layers have some level of *phase coherence* in these driven oscillations.

Let us now justify these assertions using basic ideas from thermodynamics.
10.3.1 Thermodynamics of Sustained Pulsation

Many features required to sustain stellar pulsation follow from the 1st and 2nd laws of thermodynamics.

- Let’s work in Lagrangian coordinates and assume the gas to be almost but not quite adiabatic.

- Consider a radial mass zone. By the 1st law, for a pulsation cycle the change in heat $Q$ for a mass zone is a sum of contributions from changes in internal energy $U$ and work $W$ done on its surroundings during the pulsation,

$$dQ = dU + dW.$$  

- After a complete oscillation cycle we assume that the internal energy $U$ returns to its original value so that the work done over the cycle is entirely contributed by the change in $Q$,

$$W = \oint dQ.$$

- To drive oscillations, the gas must do positive work on its surroundings (absorb heat). However, we assume the system to be nearly adiabatic, so the gas returns essentially to its original state after one cycle.

- Therefore, \textit{in zeroth order} there is no change in entropy

$$\oint dS = \oint \frac{dQ}{T} = 0.$$
• Now suppose that during the cycle we perturb the system by a small periodic variation in the temperature \( T \) of the form

\[
T = T_0 + \Delta T(t),
\]

where \( \Delta T = 0 \) at the beginning and end of the cycle.

• Then

\[
\oint dS = \oint \frac{dQ}{T} = 0 \quad \rightarrow \quad \oint \frac{dQ(t)}{T_0 + \Delta T(t)} = 0.
\]

• Assuming the variation in \( T \) to be small, we expand the denominator of the integrand to first order and obtain

\[
\oint dQ(t) (T_0 - \Delta T(t)) = 0,
\]

or, dividing by \( T_0 \) and rearranging,

\[
\oint dQ(t) = \oint \frac{\Delta T(t)}{T_0} dQ(t).
\]

• Then for the work done in one pulsation cycle

\[
\oint dQ(t) = \oint \frac{\Delta T(t)}{T_0} dQ(t) \quad \rightarrow \quad W = \oint \frac{\Delta T}{T_0} dQ.
\]
• For the cyclic integral

\[ W = \int \frac{\Delta T}{T_0} dQ \]

to give a net positive value (so that the mass zone does work on its surroundings over one cycle and can therefore drive an oscillation), we see that generally

\[ \Delta T \text{ and } dQ \text{ must have the same sign over a major part of the cycle.} \]

• That is, heat must be

1. Absorbed \((dQ > 0)\) when the temperature is increasing in the cycle \((\Delta T > 0)\), and
2. Released \((dQ < 0)\) when temperature is decreasing in the cycle \((\Delta T < 0)\).
The preceding discussion has concentrated on the behavior of a single mass zone.

- Oscillation of the entire star means that some zones may do positive work and other zones may do negative work within a pulsation cycle.

- Thus, the condition for amplifying and sustaining oscillation of the entire star is that

$$ W = \sum_i W_i = \sum_i \oint \left( \frac{\Delta T}{T_0} \right)_i dQ_i > 0, $$

where $i$ labels the mass zones of the star.

- (Strictly this sum is an integral over the continuous mass coordinate, but in practical numerical simulations the zones are normally discretized.)

We must now ask whether there are situations in stars that allow this condition to be realized.
10.3.2 The Role of Radiative Opacity

One way to favor sustained oscillations is to arrange that the opacity increases as the gas in a layer is compressed.

- Then the radiative energy outflow can be trapped more efficiently by the layer (it begins to “dam up” the outward energy flow).
- This can push it and layers above it upward until
  - the layer becomes less opaque upon expansion,
  - the trapped energy is released,
  - the layer falls back to initiate another cycle.

If a sequence of layers one above the other behaves in this way, a sustained oscillation could be set up.

- Conversely, if compressing the layer increases $T$ and thereby decreases $\kappa$, the layer allows heat to flow through it more easily than before the compression, implying that $dQ < 0$ while $T$ is increasing.
- Likewise, decompression in the 2nd part of the pulsation cycle causes $T$ to fall and $\kappa$ to increase, which traps more heat and causes $dQ$ to be positive.
- Thus heat flow works against the oscillation under these conditions and will tend to damp it.
10.3.3 Opacity and the $\kappa$-Mechanism

But normal stellar radiative opacities do not increase with compression of the gas.

- From the Kramers form
  \[ \kappa \sim \rho T^{-3.5} \]
  the opacity $\kappa$ is proportional to $\rho$ and to $T^{-3.5}$.
- Compression of a layer increases both $\rho$ and $T$.
- However, the temperature dependence is much stronger than the density dependence for $\kappa$.
- Thus a gas described by a Kramers opacity tends to experience a decrease in opacity under compression.

Hence a star exhibiting the usual opacity behavior has a built-in damping mechanism that stabilizes it against pulsations.

- This explains why most stars are not pulsating variables.
However, there is a special situation for which the opacity could be expected to increase with compression.

- If a layer contains partially ionized gas, a portion of the energy flowing into it can go into more ionization.
- Since this energy is absorbed into internal electronic excitations, it is not available to increase the temperature in the layer.
- Thus, if there is sufficient ionization during the compression portion of the pulsation cycle, the effect on the opacity of the smaller rise in temperature can be more than offset by the effect of the increase in density and compression can increase the opacity.
- Conversely, electron–ion recombination in the decompression portion of the cycle can release energy and lead to a decreased opacity.
- Then, in partial ionization zones it is possible to have a situation where a layer absorbs heat during compression when the temperature is high and releases it during expansion when the temperature is low, thereby setting the stage for a sustained oscillation.

This heat-engine mechanism for driving oscillations through ionization-dependent opacity effects is called the $\kappa$-mechanism.
10.3.4 Partial Ionization Zones and the Instability Strip

The $\kappa$-mechanism provides a possible way to drive stellar oscillations, but where do we expect the $\kappa$-mechanism to be able to operate?

- For most stars there are two significant zones of partial ionization, corresponding to the possible stages of ionization for hydrogen and helium:

1. The hydrogen ionization zone, where
   - hydrogen is ionizing ($\text{H I} \rightarrow \text{H II}$) and
   - helium is undergoing first ionization ($\text{He I} \rightarrow \text{He II}$).

   This region is broad and typically has a temperature in the range 10,000-15,000 K.

2. The helium ionization zone, where second ionization of helium ($\text{He II} \rightarrow \text{He III}$) occurs, typically at a temperature around 40,000 K.

From the preceding discussion, we may expect one or both of these ionization zones to play a role in driving the pulsations of many variable stars.
Detailed analysis indicates that

- For classical Cepheid variables (and most variables found in the instability strip) the pulsation is caused by the $\kappa$-mechanism, primarily by forcing of the fundamental mode in the helium ionization zone.

- On the other hand, the long-period red variables (large AGB stars like Mira) are thought to be driven by hydrogen ionization zones.
10.3.5 Temperature Boundaries for the Instability Strip

The (1) radial location of hydrogen and helium ionization zones in stars of particular surface temperatures, and (2) onset of convection near the surface for stars with surface temperatures that are too low, are determining factors in producing the instability strip.

- The physical radius for the hydrogen and helium partial ionization zones within a given star will depend strongly on the effective surface temperature of that star.

- For stars with higher temperatures, ionization zones will be near the surface and there will be insufficient mass in the partially-ionized layers to drive sustained oscillations.

- If the surface temperature is too low, convection in the outer layers will undermine the $\kappa$-mechanism (detailed simulations show that convection interferes with the trapping effect and thus damps stellar pulsations).

- This suggests an optimal range of surface temperatures for which

  1. the ionization zones are deep enough to drive sustained oscillations by coupling to the fundamental and overtones of the characteristic vibrational frequencies ($\rightarrow$ higher-temperature end of the optimal range),

  2. but for which the convection is not strong enough to invalidate the mechanism ($\rightarrow$ lower-temperature end of the optimal range).

Thus, pulsating variables should be found in localized regions of the HR diagram.
10.3.6 Cepheid Variables and the Helium Ionization Zone

In Fig. 10.2 opacities expected for Cepheid variable stars are plotted as a function of temperature and pressure.

- Shaded regions correspond to conditions expected to damp oscillations and lighter regions represent conditions in which the opacity increases sufficiently with increased pressure to favor the $\kappa$-mechanism.

- The dashed line indicates the relationship between $T$ and $P$ expected for a $7M_\odot$ Cepheid variable.

- The helium ionization region crossed by the dashed line near $\log T = 4.6$ is thought to be the primary driver of classical Cepheid oscillations.
10.3.7 Cepheid Variables and the Hydrogen Ionization Zone

Helium ionization zones are primarily responsible for driving pulsations within the instability strip.

- However, the hydrogen ionization zones at lower temperature nearer the surface also play a (more subtle) role in the pulsation for stars like Cepheid variables and RR Lyrae stars.

- From the figure, maximum luminosity for a Cepheid is shifted systematically later relative to minimum radius in the pulsation (maximum brightness for a Cepheid is correlated with maximum surface temperature, not maximum radius).
• This is called the phase lag, and it is thought to be caused by oscillation of the hydrogen ionization zone toward and away from the surface.

• Simulations indicate that at minimum radius the luminosity at the base of the hydrogen ionization zone is maximum,
  – but this luminosity is effectively delayed in reaching the surface because of opacity in the hydrogen ionization zone
  – thus the time of maximum surface luminosity occurs after the time of minimum radius.
10.4 The $\varepsilon$-Mechanism and Stability of Massive Stars

Before the $\kappa$-mechanism was proposed it was suggested that stellar pulsations could be driven by variations in the thermonuclear energy production caused by radial oscillations.

- This was called the $\varepsilon$-mechanism.

- Just as oscillations can be driven by the $\kappa$-mechanism if opacity increases upon contraction, the $\varepsilon$-mechanism can enhance oscillations if energy production increases upon contraction (a condition that is usually satisfied).

- Although oscillations can alter the thermonuclear energy production by causing density and temperature variations, this is of importance only in the more central regions of the star where energy production is taking place.

- The problem then is that in the central regions the amplitudes of fundamental modes and overtones are small, making it difficult for changes there to drive oscillations strongly enough to sustain them.

- Thus the $\varepsilon$-mechanism is not likely to be significant for most variable stars.

- However, it is thought that it may be important for the stability of very massive stars (of order $100 M_\odot$), where oscillations coupled to variations in energy production deep in the star may generate pulsations causing the star to shed surface layers.
10.5 Non-Radial Pulsation

For the variable stars in Table 10.1 that are labeled NR, the mode of pulsation is not spherically symmetric. The corresponding oscillations are called non-radial modes.

- Stars exhibiting non-radial pulsation include the $\delta$ Scuti stars, $\beta$ Cephei stars, and ZZ Ceti stars.

- In addition, although our own Sun is not presently classified as a variable star (it presumably will become variable after it leave the main sequence and passes through the instability strip in the HR diagram), it undergoes weak non-radial pulsations that are the target of the helioseismology observations described earlier.

- Such non-radial pulsations are somewhat beyond the scope of our present discussion.