Chapter 1

Some Properties of Stars

This chapter is assigned reading for review.
CHAPTER 1. SOME PROPERTIES OF STARS
Chapter 2

The Hertzsprung–Russell Diagram

This chapter is assigned reading for review.
Our fundamental initial task in understanding astrophysics is to understand the structure of stars. At a minimum, an understanding of stars will require

- A set of equations describing the behavior of stellar matter in gravitational fields.
- A set of equations governing energy production and associated compositional changes driven by thermonuclear reactions.
- A set of equations describing how energy is transported from the energy-producing regions deep in the star to the surface.
- Equations of state that carry information about the microscopic physics of the star and that relate macroscopic thermodynamic variables to each other.
Furthermore, these sets of equations are coupled to each other in highly non-trivial ways.

**Example:** The hydrodynamics is influenced by the energy production in the thermonuclear processes and the thermonuclear processes are in turn strongly dependent on variables such as temperature and density controlled by the hydrodynamical evolution.

- The full problem will correspond to a set of coupled, non-linear, partial differential equations that can only be solved by large-scale numerical computation.
- In many cases simplifying assumptions can be made that allow simpler solutions illustrating many basic stellar features.
- We begin the discussion by considering typical equations of state in this chapter.

**Equation of State:** A relationship among thermodynamic variables for a system that

- Carries information beyond what is known about the system on purely thermodynamic grounds.
- Often based on detailed microscopic structure input from nuclear, atomic, or particle physics.
An equation of state is of the general form

\[ P = P(T, \rho, X_i, \ldots), \]

where \( P \) is the pressure, \( T \) is the temperature, \( \rho \) is the density, the \( X_i \) are concentrations variables, and so on. The preceding equation is intended to be highly schematic at this point:

- An equation of state can take many forms.
- It need not even be specified analytically.

**Example:** Equations of state for large-scale astrophysics simulations may be specified by direct interpolation in tables that have been constructed numerically.

- Fortunately, for many (not all!) applications in astrophysics, relatively simple equations of state suffice.
3.1 The Pressure Integral

Except possibly at extremely high densities, we are primarily concerned with *equations of states for gases* in astrophysics.

If *quantum effects can be neglected* the pressure in a gas may be expressed in terms of the *pressure integral* derived in Exercise 3.6:

\[ P = \frac{1}{3} \int_{0}^{\infty} vp_n(p) \, dp, \]

where

- \( v \) is the velocity,
- \( p \) is the momentum,
- \( n(p) \) is the number density of particles with momentum in the interval \( p \) to \( p + dp \).

This formula represents a very general result that can be shown to be valid for gas particles with *any velocity, up to and including* \( v = c \) (see Exercise 3.18).
3.2. **IDEAL GASES**

3.2 Ideal Gases

If the particles in a gas may be treated as *points that interact weakly enough* the gas obeys the *ideal gas equation of state*, which may be expressed in a variety of equivalent forms:

\[
P = nkT = \frac{N}{V}kT = \frac{Nm_H}{V}RT = \rho \frac{kT}{\mu},
\]

where

- \( P \) is the pressure,
- \( n \) is the number density of gas particles,
- \( V \) is the volume,
- \( N = nV \) is the number of particles contained in a volume \( V \),
- the Boltzmann constant is \( k \) and the temperature is \( T \),
- the number of moles in the gas volume \( V \) is \( Nm_H \),
- the universal gas constant is \( R = kN_A \) (where Avogadro’s number is \( N_A = m_H^{-1} \), with \( m_H \) the atomic mass unit),
- \( \mu = \rho / nm_H \) is the average molecular weight for the gas particles and the mass density is \( \rho \).
The ideal gas equation follows from the general pressure integral

\[
P = \frac{1}{3} \int_0^\infty v p n(p) dp,
\]
evaluated specifically for a Maxwellian velocity distribution,

\[
n(p)dp = \frac{4\pi n p^2 dp}{(2\pi mkT)^{3/2}} e^{-p^2/2mkT},
\]
(see Exercise 3.7). The Maxwell velocity distribution for hydrogen gas at various temperatures is illustrated in Fig. 3.1.
For an ideal gas the internal energy $U$ is given by

$$U = \int_0^T C_V(T) \, dT,$$

with the heat capacity at constant volume $C_V(T)$ is

$$C_V(T) = \left( \frac{\partial U}{\partial T} \right)_V = T \left( \frac{\partial S}{\partial T} \right)_V,$$

where $S$ is the entropy and the first law of thermodynamics,

$$dU = \delta Q - PdV = TdS - PdV,$$

has been used. The energy density $u$ is given by

$$u = \frac{U}{V} = \frac{1}{V} \int_0^T C_V \, dT,$$

and the specific energy (energy density per unit mass) is

$$\varepsilon = \frac{u}{\rho} = \frac{U}{\rho V} = \frac{1}{\rho V} \int_0^T C_V \, dT.$$

For the special case of a monatomic, nonrelativistic ideal gas,

$$C_V = \frac{3}{2} Nk \quad U = C_V T = \frac{3}{2} NkT.$$
Expressing the internal energy in differential form,

\[ U = \int_{0}^{T} C_V(T) \, dT \quad \rightarrow \quad dU = C_V(T) \, dT, \]

introducing the heat capacity at constant pressure,

\[ C_P = \left( \frac{\partial U}{\partial T} \right)_P = T \left( \frac{\partial S}{\partial T} \right)_P, \]

and using the first law

\[ dU = T \, dS - P \, dV, \]

we have for an ideal gas (see Exercise 3.1)

\[ C_P = C_V + Nk. \]
The adiabatic index $\gamma$ is defined by

$$\gamma \equiv \frac{C_P}{C_V},$$

- For an ideal gas the heat capacities are independent of temperature and if the gas is monatomic,

$$\gamma = \frac{C_P}{C_V} = \frac{C_V + Nk}{C_V} = \frac{\frac{3}{2}Nk + Nk}{\frac{3}{2}Nk} = \frac{5}{3}.$$  

- $\gamma$ for an ideal gas is directly related to the number of degrees of freedom for each particle (Ch. 6).

- The relationship between the pressure $P$ and energy density $u$ for an ideal gas may be expressed in terms of $\gamma$ (Exercise 3.21):

$$P = (\gamma - 1)u.$$  

- This equation may be used to define an effective adiabatic index $\gamma$ for the general case, but only in the ideal gas limit is $\gamma$ given by the ratio $C_P/C_V$.

- The adiabatic speed of sound in an ideal gas is

$$v_s = \sqrt{\frac{\gamma P}{\rho}},$$

(Exercise 3.10.) where $\rho$ is density and $P$ is pressure.
3.3 Average Molecular Weights in the Gas

We are concerned with gases consisting of more than one atomic species that may be partially or totally ionized.

- For example, the gas in a star may contain
  - hydrogen atoms and ions,
  - helium atoms and ions,
  - various heavier elements as atoms or ions,
  - the electrons produced by the ionization.

- In many cases we can treat these mixtures as a single gas with an effective molecular weight.

**Example:** If the density is low, a mixture of
  - hydrogen ions,
  - fully-ionized helium ions,
  - electrons produced by the ionization

will behave as three ideal gases, each contributing a partial pressure to the total pressure (Dalton’s law of partial pressures).

Then we can treat the system as *a gas of a single kind of fictitious particle having an effective molecular weight* representing the relative contributions of each individual gas to the system properties.
3.3. AVERAGE MOLECULAR WEIGHTS IN THE GAS

3.3.1 Concentration Variables

The mass density $\rho_i$ of a species $i$ is given by

$$\rho_i = n_i A_i m_H = n_i \frac{A_i}{N_A},$$

where

- $A_i$ is the atomic mass number,
- $m_H$ is the atomic mass unit,
- $n_i$ is the number density,
- $N_A$ is Avogadro’s number.

Let us introduce the mass fraction $X_i$ by

$$X_i \equiv \frac{\rho_i}{\rho} = \frac{n_i A_i m_H}{\rho} = \frac{n_i A_i}{\rho N_A},$$

where

- $\rho$ is the total mass density.
- The label $i$ may refer to ions, atoms, or molecules.
- The mass fractions sum to unity: $\sum X_i = 1$.

We also will use the mole fraction or abundance $Y_i$,

$$Y_i \equiv \frac{X_i}{A_i} = \frac{n_i}{\rho N_A}.$$

Generally, the sum of the $Y_i$ will not be unity.
3.3.2 Total Mean Molecular Weight

If the energy density and momentum carried by radiation is ignored the average mass of a gas particle (atoms, ions, and electrons) is given by

$$\mu = \left( \sum_i (1 + y_i Z_i) Y_i \right)^{-1},$$

- The sum is over isotopic species $i$.
- $y_i$ is the fractional ionization of the species $i$ ($y_i = 0$ for no ionization and $y_i = 1$ if the species $i$ is completely ionized).
- $Z_i$ is the atomic number for isotopic species $i$.

In very hot stars the momentum and energy density carried by photons is non-trivial and we will see later that this will further modify the effective mean molecular weight of the gas.

With this formalism we have replaced the actual gas (a mixture of electrons and different atomic, possibly molecular, and ionic species) with a gas containing a single kind of fictitious particle having an effective mass $\mu$ (often termed the mean molecular weight) that is given by the preceding equation.
Example 3.1

The composition of many white dwarfs may be approximated by a completely ionized gas consisting of equal parts $^{12}\text{C}$ and $^{16}\text{O}$ by mass. The mass fractions are $X_{^{12}\text{C}} = X_{^{16}\text{O}} = 0.5$, so the abundances are

\[
Y_{^{12}\text{C}} = \frac{X_{^{12}\text{C}}}{12} = \frac{0.5}{12} = 0.04167 \quad Y_{^{16}\text{O}} = \frac{X_{^{16}\text{O}}}{16} = \frac{0.5}{16} = 0.03125.
\]

Therefore,

\[
\mu = \frac{1}{\sum_i (1 + y_i Z_i) Y_i} = \frac{1}{(1 + 6)0.04167 + (1 + 8)0.03125} = 1.745 \text{ amu}
\]

if we assume complete ionization ($y_i = 1$).
3.3.3 Common Notation

A common shorthand notation is to define

\[ X \equiv X_{\text{hydrogen}} \quad Y \equiv X_{\text{helium}} \quad Z \equiv X_{\text{metals}} \]

where “metals” refers to the sum of all elements heavier than helium and \( X + Y + Z = 1 \).

For a typical Pop I star just entering the main sequence (termed a Zero-Age Main Sequence or ZAMS star) we find

\[ X \simeq 0.7 \quad Y \simeq 0.3 \quad Z \simeq 0.03 \]

The metal concentration \( Z \) will be less in Pop II stars.
An ideal gas equation of state (with an effective mean molecular weight $\mu$) is often a very realistic approximation for many astrophysical processes.

- But there are other possible equations of state that play an important role in particular contexts.
- One example is a *polytropic equation of state*.
- Generally, a *polytropic process* is defined by the requirement

$$\frac{\delta Q}{\delta T} = c$$

where

- $Q$ is the heat,
- $T$ is the temperature,
- and $c$ is a constant.
Some important properties of polytropes are explored in Exercise 3.11. For example,

- From the first law of thermodynamics and
  \[ \frac{\delta Q}{\delta T} = c \]
  we may prove that polytropic processes in ideal gases obey
  \[ \frac{dT}{T} = (1 - \gamma) \frac{dV}{V} \]
  where the polytropic \( \gamma \) is defined by
  \[ \gamma \equiv \frac{C_P - c}{C_V - c}. \]
  The polytropic \( \gamma \) reduces to the ideal gas adiabatic parameter \( \gamma \) only if \( c = 0 \).

- You may verify by substitution that the differential equation
  \[ \frac{dT}{T} = (1 - \gamma) \frac{dV}{V} \]
  has three classes of solutions:
  \[ PV^\gamma = C_1 \quad P^{1-\gamma}T^\gamma = C_2 \quad TV^{\gamma-1} = C_3, \]
  (polytropic equations of state) with \( C_n \) constants.
The most common form of a polytropic equation of state employed in astrophysics is

\[ P(r) = K \rho^\gamma(r) = K \rho^{1+1/n}(r), \]

where the polytropic index \( n \) is parameterized by

\[ n = \frac{1}{\gamma - 1} \]

with the parameter \( \gamma \) not necessarily defined by the ideal gas expression.

Note: This is of the form \( PV^\gamma = C_1 \), since the density \( \rho \propto 1/V \).

1. The polytropic approximation implies physically that the pressure is independent of the temperature, depending only on density and composition.

2. This approximation often makes solution of the equations for stellar structure easier because it decouples the differential equations describing hydrostatic equilibrium from those governing energy transfer and the temperature gradients.
Examples where polytropes are appropriate:

1. For a completely ionized star, fully mixed by convection with negligible radiation pressure,

   \[ P = K \rho^{5/3}, \]

   which corresponds to a polytrope with \( \gamma = 5/3 \) and \( n = 3/2 \). The parameter \( K \) is constant for a given star, but can differ from star to star.

2. For a completely degenerate gas of nonrelativistic fermions (defined below)

   \[ P = K \rho^{5/3}, \]

   again corresponding to a polytrope with \( \gamma = 5/3 \) and \( n = 3/2 \), but now \( K \) is fixed by fundamental constants.

3. For a completely degenerate gas of ultrarelativistic fermions (defined below)

   \[ P = K \rho^{4/3}, \]

   corresponding to a polytrope with \( \gamma = 4/3 \) and \( n = 3 \), with \( K \) again fixed by fundamental constants.

These three polytropic equations of state will be relevant for homogenous stars that are completely mixed by convection, white dwarfs, and neutron stars, respectively.
In terms of the heat $Q$ and the entropy $S$, adiabatic processes are defined by the condition that

$$\delta Q = T \delta S = 0,$$

- From the first law,

$$dU = \delta Q - PdV = T \delta S - PdV,$$

we see that in adiabatic processes the change in internal energy

$$dU = -PdV$$

comes only from $PdV$ work.

- Since adiabatic processes do not exchange heat with their environment, they are fully reversible ($dS = 0$).

- Most phenomena in astrophysics are not adiabatic but many are at least approximately so.
It is standard practice to introduce three \textit{adiabatic exponents} $\Gamma_1, \Gamma_2$, and $\Gamma_3$ through

\[
\Gamma_1 \equiv \left( \frac{\partial \ln P}{\partial \ln \rho} \right)_S, \quad \frac{\Gamma_2}{\Gamma_2 - 1} \equiv \left( \frac{\partial \ln P}{\partial \ln T} \right)_S, \quad \Gamma_3 - 1 \equiv \left( \frac{\partial \ln T}{\partial \ln \rho} \right)_S
\]

where the subscripts $S$ remind us that adiabatic processes correspond to constant entropy, and where the logarithmic derivatives are equivalent to $\partial \ln A = \partial A / A$. This implies equations of state having one of the three forms

\[
P V^{\Gamma_1} = c_1 \quad P^{1 - \Gamma_2 T^{\Gamma_2}} = c_2 \quad T V^{\Gamma_3 - 1} = c_3
\]

where the $c_n$ are constants.

From the above definitions,

\[
\Gamma_1 (\Gamma_2 - 1) = \Gamma_2 (\Gamma_3 - 1),
\]

so the $\Gamma_i$ are not independent.

We also introduce,

\[
\nabla_{\text{ad}} \equiv \frac{\Gamma_2 - 1}{\Gamma_2},
\]

which will play a role in our later discussion of convective gas motion in stars.
For the special case of ideal gases

\[
\Gamma_1 \equiv \left( \frac{\partial \ln P}{\partial \ln \rho} \right)_S \quad \frac{\Gamma_2}{\Gamma_2 - 1} \equiv \left( \frac{\partial \ln P}{\partial \ln T} \right)_S
\]

\[\Gamma_3 - 1 \equiv \left( \frac{\partial \ln T}{\partial \ln \rho} \right)_S\]

are equal and equivalent to the ideal gas \( \gamma \) (Exercise 3.13),

\[\Gamma_1 = \Gamma_2 = \Gamma_3 = \gamma \quad \text{(ideal gas)}\]

But for more general equations of state \( \Gamma_1, \Gamma_2, \) and \( \Gamma_3 \)
are distinct and carry information emphasizing different aspects of the gas thermodynamics:

1. Because it specifies how changes in pressure are related to changes in density, \( \Gamma_1 \) enters into dynamical properties of the gas such as the speed of sound.

2. \( \Gamma_2 \), and the quantity \( \nabla_{\text{ad}} \) defined through it, relate changes in pressure to changes in temperature, which is important in understanding convective gas motion.

3. \( \Gamma_3 \) depends on the derivative of temperature with respect to density, which influences the response of the gas to compression.

An example of these differences is given below for a mixture of ideal gas and photons in the adiabatic limit.
3.6 Quantum Mechanics and Equations of State

Stellar equations of state reflect microscopic properties of the gas in stars.

- At low densities this gas tends to behave classically,
- But the correct microscopic theory of matter is quantum mechanics and at higher densities a quantum description becomes essential to an accurate treatment.  

The requisite physics can be understood conceptually in terms of four basic ideas.

1. **deBroglie Wavelength:** The foundation of a quantum description of matter is particle–wave duality:
   - A particle at the microscopic level takes on wave properties characterized by a deBroglie wavelength \( \lambda = \frac{h}{p} \), where \( p \) is the momentum and \( h \) is Planck’s constant.
   - Thus in quantum mechanics the location of a particle becomes fuzzy, spread out over an interval comparable to the deBroglie wavelength.

2. **Heisenberg Uncertainty Principle:** The uncertainty principle quantifies the fuzziness of particle–wave duality, requiring that
   - \( \Delta p \cdot \Delta x \geq \hbar \), where \( \Delta p \) is the uncertainty in momentum, \( \Delta x \) is the uncertainty in position, and \( \hbar = \frac{h}{2\pi} \).
   - \( \Delta E \cdot \Delta t \geq \hbar \), where \( \Delta E \) is the uncertainty in energy and \( \Delta t \) is the uncertainty in the time over which the energy is measured.
3. Quantum Statistics: All elementary particles may be classified as either fermion or bosons. These classifications have to do with how aggregates of elementary particles behave.

- Fermions (such as electrons, or neutrons and protons if we neglect their internal quark and gluon structure) obey Fermi–Dirac statistics.
  - The most notable consequence is the Pauli exclusion principle: no two fermions can have an identical set of quantum numbers.
  - All elementary particles of half-integer spin are fermions.

- Bosons (photons are the most important example for our purposes) obey Bose–Einstein statistics.
  - Unlike for fermions, there is no restriction on how many bosons can occupy the same quantum state.
  - All elementary particles of integer spin are bosons.

- Matter is made from fermions (electrons, protons, neutrons, ...).
- Forces between particles are mediated by the exchange of bosons (for example, the electromagnetic force results from an exchange of photons between charged particles).
4. **Degeneracy**: The exclusion principle implies that in a many-fermion system *each fermion must be in a different quantum state*.

- Thus the lowest-energy state results from filling energy levels from the bottom up.

- *Degenerate matter* corresponds to a many-fermion state in which all the lowest energy levels are filled and all the higher ones are unoccupied.

- Degenerate matter occurs frequently at high densities and has a very unusual equation of state with a number of implications for astrophysics.
3.7 Equations of State for Degenerate Gases

Degenerate equations of state play an important role in a variety of astrophysical applications. For example,

- In white dwarf stars the electrons are highly degenerate.
- In neutron stars the neutrons are highly degenerate.

Let us look at this in a little more detail for the case of degenerate electrons.

- We first demonstrate that (as a consequence of quantum mechanics) most stars are completely ionized over much of their volume because ionization can be induced by sufficiently high pressure, even at low temperature.
- This implies the possibility of producing a (relatively) cold gas of electrons, which is the necessary condition for a degenerate electron equation of state.
3.7.1 Pressure Ionization

Consider the schematic diagram shown in Fig. 3.2, where

- Atoms occupy the darker spheres of radius $r$.
- The average spacing between atoms is represented in terms of the lighter spheres with radius $d$.
- To illustrate simply we assume for that the stellar material consists only of
  - Ions of a single species
  - Electrons produced by ionizing that species
3.7. EQUATIONS OF STATE FOR DEGENERATE GASES

- Electrons in the atoms obey Heisenberg relations of the form

\[ p \cdot \Delta x \geq \hbar. \]

- Taking an average volume per electron of \( V_0 \approx (\Delta x)^3 \), the uncertainty relation becomes

\[ p \geq \hbar/V_0^{1/3}. \]

- The uncertainty principle produces ionization when the effective volume of the atoms becomes too small to confine the electrons.

- The volume per electron \( V_0 \) and the volume per ion \( V_i \) are related by \( ZV_0 = V_i \), since there are \( Z \) electrons per ion. Thus

\[ p \geq \hbar/V_0^{1/3} \quad \rightarrow \quad p \geq \hbar Z^{1/3}/V_i^{1/3}. \]
• From atomic physics the atomic radius may be approximated by
  \[ r \simeq a_0 Z^{-1/3}, \]
  where \( a_0 = 5.3 \times 10^{-9} \text{ cm} \) is the Bohr radius.

• If the star is composed entirely of an element with atomic number \( Z \) and mass number \( A \), there are \( Z \) electrons in each sphere of radius \( d \) and the average number density of electrons is

  \[ n_e = \frac{Z}{\frac{4}{3} \pi d^3} \quad \rightarrow \quad d \simeq \left( \frac{3Z}{4\pi n_e} \right)^{1/3}. \]
• Provided that
\[ d = \left( \frac{3Z}{4\pi n_e} \right)^{1/3} < r, \]
we may expect pressure ionization, as illustrated in the following figure:

With increasing density fewer locally bound states are possible until none remain and the electrons are all ionized. Thus, sufficiently high density can cause complete ionization, even at zero temperature.
• Since there are $A$ nucleons in each volume of radius $d$ in the above figure, the mass density $\rho$ is

$$\rho = \frac{Am_H}{\frac{4}{3} \pi d^3}$$

and requiring that $d \simeq r \simeq a_0 Z^{-1/3}$ defines a critical density

$$\rho_{\text{crit}} \simeq \frac{Z Am_H}{\frac{4}{3} \pi a_0^3}.$$

• We may expect that for densities greater than this there will be almost complete pressure ionization, irrespective of the temperature.
Table 3.1: Critical pressure-ionization densities

<table>
<thead>
<tr>
<th>Element</th>
<th>((Z,A))</th>
<th>Density ((\text{g cm}^{-3}))</th>
</tr>
</thead>
<tbody>
<tr>
<td>Hydrogen</td>
<td>(1, 1)</td>
<td>3.2</td>
</tr>
<tr>
<td>Helium</td>
<td>(2, 4)</td>
<td>26</td>
</tr>
<tr>
<td>Carbon</td>
<td>(6, 12)</td>
<td>230</td>
</tr>
<tr>
<td>Oxygen</td>
<td>(8, 16)</td>
<td>410</td>
</tr>
<tr>
<td>Iron</td>
<td>(26, 56)</td>
<td>4660</td>
</tr>
</tbody>
</table>

• The density condition

\[
\rho_{\text{crit}} \approx \frac{ZAm_H}{4\pi a_0^3}.
\]

is fulfilled rather easily. Consider the case of pure hydrogen. Inserting \(Z = A = 1\) gives a critical density of 3.2 g cm\(^{-3}\), only a little larger than that of water.

• The critical pressure ionization densities for gases composed of a few representative elements are summarized in Table 3.1.

• These critical densities may be compared with typical actual densities of order

  \(\sim 100 \text{ g cm}^{-3}\) for the center of the Sun,

  \(\sim 10^6 \text{ g cm}^{-3}\) for a carbon–oxygen white dwarf,

  \(\sim 10^9 \text{ g cm}^{-3}\) for the iron core of a massive pre-supernova star.

• These considerations imply that Saha ionization equations, which are derived assuming ionization to be caused by thermal effects, are no longer reliable in the deep interior of stars.
The Saha equations would predict approximately 24% of the hydrogen in the core of the Sun to be neutral.

- However, comparison of the preceding table with properties of the solar interior indicates that the density is sufficiently high to pressure ionize hydrogen over the inner 40% of the Sun.

- Note that increased pressure favors pressure ionization but disfavors thermal ionization because of increased recombination.

Between thermal and pressure effects, the solar interior is almost entirely ionized, in contrast to what we would expect from the Saha equations alone.
3.7.2 Classical and Quantum Gases

Let us now consider the distinction between a classical gas and a quantum gas, and the corresponding implications for stellar structure.
Identical fermions are described statistically in quantum mechanics by the Fermi–Dirac distribution

\[ f(\varepsilon_p) = \frac{1}{\exp[(\varepsilon_p - \mu)/kT] + 1} \quad \text{(Fermi–Dirac)}, \]

where \( \varepsilon_p \) is given by

\[
\begin{align*}
\varepsilon_p &= mc^2 + \frac{p^2}{2m} \quad \text{(nonrelativistic)} \\
\varepsilon_p^2 &= p^2 c^2 + m^2 c^4 \quad \text{(relativistic)}
\end{align*}
\]

and the chemical potential \( \mu \) is introduced by adding to the First Law a term accounting for a possible change in the particle number \( N \):

\[ dU = TdS - PdV + \mu dN, \]

where \( T, P, \) and \( \mu \) are taken as the macroscopic thermodynamical variables for the gas and \( S \) is the entropy. Identical bosons are described by the Bose–Einstein distribution

\[ f(\varepsilon_p) = \frac{1}{\exp[(\varepsilon_p - \mu)/kT] - 1} \quad \text{(Bose–Einstein)}. \]
We shall consider a gas to be a *quantum gas* if it is described by one of the distributions

\[ f(\varepsilon_p) = \frac{1}{\exp[(\varepsilon_p - \mu)/kT] + 1} \]  
(Fermi–Dirac),

\[ f(\varepsilon_p) = \frac{1}{\exp[(\varepsilon_p - \mu)/kT] - 1} \]  
(Bose–Einstein).

and a *classical gas* if the condition

\[ e^{(mc^2 - \mu)/kT} \gg 1 \]

is fulfilled. If the gas is classical,

- The states of lowest energy have \( \varepsilon_p \sim mc^2 \).
- For either fermions or bosons the distribution function becomes well approximated by *Maxwell–Boltzmann statistics*,

\[ f(\varepsilon_p) = e^{-(\varepsilon_p - \mu)/kT} \]  
(Maxwell–Boltzmann),

where generally \( f(\varepsilon_p) \ll 1 \) for the classical gas.

- Thus, in the classical gas
  - The lowest energy states are scarcely occupied
  - The Pauli principle plays little role
  - The gas obeys Maxwell–Boltzmann statistics
We now demonstrate that the conditions for forming a classical gas are equivalent to a constraint on the density such that the deBroglie wavelength of the particles in the gas is considerably less than the average interparticle spacing in the gas.
Non-Relativistic Classical and Quantum Gases:

Let us introduce a critical (number) density variable \( n_Q \)

\[
n_Q \equiv \left( \frac{2\pi mkT}{\hbar^2} \right)^{3/2} = \frac{(2\pi)^{3/2}}{\lambda^3},
\]

where the deBroglie wavelength \( \lambda \) for non-relativistic particles is given by

\[
\lambda = \frac{\hbar}{p} \simeq \left( \frac{\hbar^2}{mkT} \right)^{1/2}.
\]

The number of particles in the gas is

\[
N = \int_0^\infty f(\varepsilon_p)g(p)\,dp,
\]

where the integration measure may be expressed as

\[
g(p)dp = g_s V \frac{\hbar^3}{4\pi p^2} dp,
\]

with \( p \) the momentum and \( g_s = 2j + 1 = 2 \) the spin degeneracy factor for electrons. Substituting the Maxwell–Boltzmann distribution with a nonrelativistic energy,

\[
f(\varepsilon_p) = e^{-(\varepsilon_p - \mu)/kT} \quad \varepsilon_p = mc^2 + \frac{p^2}{2m}
\]

integrating, and rearranging yields

\[
mc^2 - \mu = kT \ln \left( \frac{g_s n_Q}{n} \right),
\]

where the number density \( n \) is given by \( n = N/V \).
Therefore, the classical gas condition
\[ e^{(mc^2 - \mu)/kT} >> 1 \]
is, by virtue of
\[ mc^2 - \mu = kT \ln \left( \frac{g_s n_Q}{n} \right), \]
a requirement that at a given temperature \( n << n_Q \).

In a classical gas, the actual number density \( n \) is small on the scale set by the quantum density \( n_Q \).
The preceding result has an alternative interpretation:

- The average separation between particles in the gas is \( d \sim n^{-1/3} \rightarrow 1/n \sim d^3 \),
- The condition \( n \ll n_Q \) defining a classical gas implies that \( 1/n \gg 1/n_Q \),
- Because
  \[
  n_Q = \frac{(2\pi)^{3/2}}{\lambda^3} \rightarrow n_Q \sim \lambda^{-3} \rightarrow 1/n_Q \sim \lambda^3
  \]
  the condition \( n \ll n_Q \) is equivalent to requiring that
  \[
  \frac{1}{n} \gg \frac{1}{n_Q} \rightarrow d^3 \gg \lambda^3 \rightarrow d \gg \lambda.
  \]
  
For a classical gas, the average separation between particles must be much larger than the average deBroglie wavelength \( \lambda \) for particles in the gas.

- This makes sense conceptually:
  - The “quantum fuzziness” of a particle extends over a distance \( \lambda \).
  - If particles are separated on average by distances larger than \( \lambda \), quantum effects are minimized.
Illustration of classical and quantum gases.

- In the classical gas the average spacing $d$ between gas particles is much larger than their deBroglie wavelengths $\lambda$.

- In the quantum gas the interparticle spacing is comparable to or greater than the deBroglie wavelength. distribution.

The gas particles have a range of deBroglie wavelengths because they have a thermal velocity distribution.
Ultrarelativistic Classical and Quantum Gases:

Proceeding in a manner similar to that for the non-relativistic case, for ultrarelativistic particles \((v \sim c)\) the rest mass of the particle may be neglected and from

\[ \varepsilon_p \simeq kT = \sqrt{m^2c^4 + p^2c^2} \simeq \sqrt{p^2c^2} \]

and

\[ f(\varepsilon_p) = e^{-\left(\varepsilon_p - \mu\right)/kT}, \]

we obtain

\[ \mu = -kT \ln \left( \frac{g_s n_R^Q}{n} \right), \]

where the relativistic quantum critical density variable is defined by

\[ n_R^Q = 8\pi \left( \frac{kT}{hc} \right)^3. \]

Hence in the ultrarelativistic case

- The condition that the gas be a classical one is equivalent to a requirement that \(n \ll n_R^Q\).

- This again is equivalent to requiring that the de-Broglie wavelength

\[ \lambda = \frac{h}{p} \simeq \frac{hc}{kT} \]

be small compared with the average separation of particles in the gas.
### 3.7.3 Transition from Classical to Quantum Gas Behavior

We conclude from the preceding results that at high enough density the gas behaves as a quantum gas. Notice from

\[
n_Q \equiv \left( \frac{2\pi mkT}{\hbar^2} \right)^{3/2} = \frac{(2\pi)^{3/2}}{\lambda^3},
\]

that with increasing gas density

- The least massive particles in the gas will be most prone to a deviation from classical behavior because the critical density is proportional to \( m^{3/2} \).
- Thus photons, neutrinos, and electrons are most susceptible to such effects.
- The massless photons *never behave as a classical gas* and the neutrinos (also massless or nearly so) interact so weakly with matter that they leave the star unimpeded when they are produced.
- It follows that in normal stellar environments it is the *electrons that are most susceptible* to a transition from classical to quantum gas behavior.

---

\[a\] An exception to this statement occurs for core-collapse supernovae, where the densities and temperatures become high enough to trap neutrinos for significant times relative to the dynamical timescales for the problem.
Presently in the center of the Sun

- The average number density for electrons is \( n \sim 6 \times 10^{29} \text{ m}^{-3} \)

- For a temperature of \( 16 \times 10^6 \text{ K} \) the nonrelativistic critical quantum density is \( n_Q \sim 3 \times 10^{31} \text{ m}^{-3} \).

- Thus, \( n/n_Q \sim 0.02 \) and electrons in the Sun are reasonably well approximated by a dilute classical gas.

However, the core of the Sun, as for all stars, will contract late in its life as its nuclear fuel is exhausted.

- The approximate relationship between a star’s temperature \( T \) and radius \( R \) is \( kT \simeq 1/R \), which implies that

  \[
  n_Q \equiv \left( \frac{2\pi mkT}{\hbar^2} \right)^{3/2} = \left( \frac{2\pi m}{\hbar^2 R} \right)^{3/2} \simeq R^{-3/2}
  \]

  and, because the actual number density behaves as \( n \sim R^{-3} \), that

  \[
  \frac{n}{n_Q} \simeq \frac{R^{-3}}{R^{-3/2}} \simeq R^{-3/2}.
  \]

As the core of the Sun contracts later in its evolution, eventually the electrons in it will begin to behave as a quantum rather than classical gas.
3.7.4 The Degenerate Electron Gas

From the definition

\[ n_Q \equiv \left( \frac{2\pi mkT}{h^2} \right)^{3/2} \]

the quantum gas condition \( n \gg n_Q \) is equivalent to a temperature constraint,

\[ kT \ll \frac{h^2 n^{2/3}}{2\pi m}. \]

A quantum gas is a cold gas, but *cold on a temperature scale set by the right side of the preceding equation.*

- If the number density \( n \) is high enough, this equation indicates that the gas could be “cold” and still have a temperature of billions of degrees!
- The precise quantum mechanical meaning of a cold electron gas is that the electrons are all concentrated in the lowest available quantum states.
- We say that such a gas is *degenerate.*
- Degenerate gases have much in common with the metallic state in condensed matter.

*Example:* Ideal gases have very low thermal conductivity (that is why down parkas keep you warm). Degenerate electron gases, as for metals, are very good thermal conductors.
In the limit that the temperature may be neglected, the Fermi–Dirac distribution becomes a step function in energy space (see figure above),

\[
f_f(\varepsilon_p) = \frac{1}{e^{(\varepsilon_p - \mu)/kT} + 1} \xrightarrow{T \to 0} \begin{cases} f(\varepsilon_p) = 1 & \varepsilon_p \leq \varepsilon_f \\ f(\varepsilon_p) = 0 & \varepsilon_p > \varepsilon_f \end{cases}
\]

where

- The value of the chemical potential \( \mu \) at zero temperature is denoted by \( \varepsilon_f \) and is termed the fermi energy.
- The corresponding value of the momentum is denoted by \( p_f \) and is termed the fermi momentum.
- Thus, the fermi energy gives the energy of the highest occupied state in the degenerate fermi gas.
The density of states as a function of momentum $p$ is given by $g(p)$ in

$$g(p) dp = g_s \frac{V}{\hbar^3} 4\pi p^2 dp,$$

and the number of electrons in the degenerate gas at zero temperature is just the number of states with momentum less than the fermi momentum $p_f$,

$$N = \int_0^{p_f} g(p) \, dp$$

$$= 4\pi V g_s \frac{\hbar}{h^3} \int_0^{p_f} p^2 \, dp$$

$$= \frac{8\pi V}{3h^3} p_f^3,$$

where $g_s = 2$ has been used for electrons. Solving for the fermi momentum $p_f$ and introducing the number density $n = N/V$, we find that the fermi momentum is determined completely by the electron number density

$$p_f = \left( \frac{3h^3}{8\pi} \cdot \frac{N}{V} \right)^{1/3} = \left( \frac{3n}{8\pi} \right)^{1/3} \hbar.$$

For a number density $n$ the interparticle spacing is $\sim n^{-1/3}$, so this result implies that the deBroglie wavelength of an electron at the fermi surface, $\lambda = \hbar / p_f \sim n^{-1/3}$, is comparable to the average spacing between electrons.
We may construct the equation of state for the degenerate electron gas by evaluating the internal energy of the gas. Let us first do this in the nonrelativistic and then in the ultrarelativistic limits for degenerate electrons.
Nonrelativistic Degenerate Electrons:

In the nonrelativistic limit $p_f << mc$, which implies (since $p_f \sim \hbar n^{1/3}$) that

$$n << \left( \frac{mc}{\hbar} \right)^3 = \left( \frac{1}{\lambda_c} \right)^3,$$

where $\lambda_c \equiv \hbar / mc$ is the Compton wavelength for an electron. In this limit the internal energy for the degenerate electron gas (for which $f(\epsilon_p)$ is a step function) is

$$U = \int_0^\infty \epsilon_p f(\epsilon_p) g(p) dp \simeq Nmc^2 + \frac{3N}{10m} p_r^2,$$

where

$$\epsilon_p = mc^2 + \frac{p^2}{2m} \quad g(p) = g_s \frac{V}{h^3} 4\pi p^2 \quad N = \frac{8\pi V}{3h^3} p_r^3$$

and $g_s = 2$ have been used. For a nonrelativistic gas the pressure is given by $\frac{2}{3}$ of the kinetic energy density; identifying the second term of

$$U = Nmc^2 + \frac{3N}{10m} p_r^2,$$

divided by the volume $V$ as the kinetic energy density,

$$P = \frac{2}{3} \times \text{(kinetic energy density)} = \frac{2}{3} \left( \frac{N 3 p_r^2}{V 10m} \right)$$

$$= \frac{n p_r^2}{5m} = \frac{h^2}{5m} \left( \frac{3}{8\pi} \right)^{2/3} n^{5/3} \quad (\gamma = \frac{5}{3} \text{ polytrope}),$$

where $n = N/V$ and $p_r = (3n/8\pi)^{1/3} \hbar$ have been used.
Example: For low-mass white dwarfs with \( \rho \lesssim 10^6 \text{ g cm}^{-3} \) the electrons are nonrelativistic and the electron pressure is given by the \( n = \frac{3}{2} \) polytrope implied by the preceding equation,

\[
P_e = \frac{\hbar^2}{5m} \left( \frac{3}{8\pi} \right)^{2/3} \left( \frac{\rho}{m_p \mu_e} \right)^{5/3}
\]

with the mean molecular weight \( \mu_e \) defined through \( n_e = \frac{\rho}{m_p \mu_e} \). As noted earlier, the constant \( K \) in the polytropic form

\[
P = K \rho^\gamma (r)
\]

is fixed by fundamental constants.
Ultrarelativistic Degenerate Electrons:

For ultrarelativistic electrons, \( n \gg n_Q^R \) implies that \( n \gg (mc/h)^3 \). Utilizing the ultrarelativistic limit \( \varepsilon_p = pc \) and

\[
g(p) = g_s \frac{V}{h^3} 4\pi p^2
\]

the internal energy is

\[
U = \int_0^\infty \varepsilon_p f(\varepsilon_p) g(p) \, dp
\]

\[
\simeq \frac{8\pi V c}{h^3} \int_0^{p_f} p^3 \, dp = \frac{3}{4} N cp_f.
\]

For an ultrarelativistic gas the pressure is \( \frac{1}{3} \) of the kinetic energy density. Identifying the kinetic energy density as \( U = \frac{3}{4} N cp_f \) divided by the volume \( V \), for ultrarelativistic particles

\[
P = \frac{1}{3} \times (\text{kinetic energy density})
\]

\[
= \frac{1}{3} \times \left( \frac{3}{4} cp_f \right) = \frac{hc}{4} \left( \frac{3}{8\pi} \right)^{1/3} n^{4/3},
\]

where \( n = N/V \) and we have used \( p_f = (3n/8\pi)^{1/3} \).
Example: For higher-mass white dwarfs having \( \rho \gtrsim 10^6 \) g cm\(^{-3}\) the electrons are highly relativistic and the corresponding degenerate equation of state takes the form implied by the preceding equation,

\[
P_e = \frac{hc}{4} \left( \frac{3}{8\pi} \right)^{1/3} \left( \frac{\rho}{m_p\mu_e} \right)^{4/3}
\]

which is a polytrope with \( n = 3 \). As in the nonrelativistic case, we see that the constant \( K \) multiplying \( \rho^{4/3} \) is fixed by fundamental considerations.
3.7.5 Summary: High Gas Density and Stellar Structure

This preceding discussion implies that increasing the density can have a large impact on the structure of stars. Generally, we may identify several important consequences of high densities in stellar environments:

- An increase in the gas density above a critical amount enhances the probability for pressure ionization, thereby creating a gas of electrons and ions irrespective of possible thermal ionization.

- An increase in the gas density, by uncertainty principle arguments, increases the average momentum of gas particles and makes them more relativistic.

- An increased density raises the fermi momentum. This, for example, influences the weak interaction processes that can take place in the star.

- An increase in the gas density decreases the interparticle spacing relative to the average deBroglie wavelength, making it more likely that the least massive particles in the system make a transition from classical to degenerate quantum gas behavior.
3.7. EQUATIONS OF STATE FOR DEGENERATE GASES

- Increased density enhances the strength of the gravitational field and makes it more difficult to maintain stability of the star against gravitational collapse. Higher density also makes it more likely that general relativistic corrections to Newtonian gravitation become important.

- Higher density (often implying higher temperature) tends to change the rates of thermonuclear reactions and to alter the opacity of the stellar material to radiation.

  1. The former changes the rate of energy production;
  2. the latter changes the efficiency of how that energy is transported in the star.

Both can have large consequences for stellar structure and evolution.

These consequences of increased density have large implications for the structure and evolution of stars because all stars are expected to go through late evolutionary stages that may dramatically increase their central densities.
From the preceding, a gas can have a pressure that is of purely quantum-mechanical origin, independent of its temperature.

- Assume pressure dominated by non-relativistic electrons and drop factors like $\frac{1}{2}$. For an ideal gas the average energy of an electron is $E \sim kT = \frac{1}{2}m_e v^2$, so electrons have a velocity
  \[ v_{\text{thermal}} \simeq \left( \frac{kT}{m_e} \right)^{1/2} \]
of purely thermal origin.

- But even at zero temperature the electrons have a velocity $v_{\text{QM}}$ implied by the uncertainty principle, since
  \[ p \sim \Delta p \sim \frac{\hbar}{\Delta x} \sim \hbar n_e^{1/3} \rightarrow v_{\text{QM}} \simeq \frac{p}{m_e} \simeq \frac{\hbar n_e^{1/3}}{m_e}. \]

- Thus, the average velocity of particles in the gas has two contributions,
  1. one from the finite temperature and
  2. one from quantum mechanics,

  with the thermal contribution vanishing identically at zero temperature.

- The pressure contributed by the thermal motion is
  \[ P_{\text{thermal}} = n_e kT = n_e m_e v_{\text{thermal}}^2 \]
  and the pressure contributed by quantum mechanics is (up to some constant factors)
  \[ P_{\text{QM}} \simeq \frac{\hbar^2}{m_e n_e^{5/3}} = n_e m_e \left( \frac{\hbar n_e^{1/3}}{m_e} \right)^2 = n_e m_e v_{\text{QM}}^2 \]

- A degenerate gas is one for which $P_{\text{QM}} \gg P_{\text{thermal}}$. 
• The thermal pressure is proportional to $T$ and density, but the quantum-mechanical pressure is independent of $T$ and proportional to a power of the density:

$$P_{\text{thermal}} = n_e kT \quad P_{\text{QM}} \approx \frac{\hbar^2}{m_e} n_e^{5/3}.$$ 

• Therefore, degeneracy is favored in low-temperature, dense gases, and a gas can have a high pressure of purely uncertainty-principle origin, even at $T = 0$.

• Furthermore, changing $T$ in a highly-degenerate gas will have little effect on the pressure, which is dominated by a term independent of $T$ (as long as changing the temperature does not significantly change the degeneracy of the gas).

All of these properties have profound consequences for stellar structure and stellar evolution when high densities are encountered.
3.8 Equation of State for Radiation

We may view electromagnetic radiation in stars as an ultrarelativistic gas of massless bosons. The equation of state associated with radiation follows from substituting the blackbody (Planck) frequency distribution

\[ n(\nu)d\nu = \frac{8\pi\nu^2 d\nu}{c^3(e^{\nu kT} - 1)}, \]

into the pressure equation

\[ P = \frac{1}{3} \int_0^\infty \nu n(p) dp, \]

and evaluating the integral (see Exercise 3.8). This yields for the radiation pressure,

\[ P_{\text{rad}} = \frac{4}{3} aT^4, \]

where \( a \) is the radiation density constant. The corresponding energy density of the radiation field is

\[ u_{\text{rad}} = a T^4 = 3P_{\text{rad}}. \]
Gravitational Stability and Adiabatic Exponents for Radiation:

You are asked to show in Exercise 3.14 that the adiabatic exponents $\Gamma_1$, $\Gamma_2$, and $\Gamma_3$ for a pure radiation field are all equal to $\frac{4}{3}$.

- As we shall see in more detail later, an adiabatic exponent less than $\frac{4}{3}$ generally implies an *instability against gravitational collapse*.
- Therefore, admixtures of radiation contributions (more generally, of any relativistic component) to pressure often signal *decreased gravitational stability for a gas*. 
For a simple stellar model, it is often a good starting point to assume

- An ideal gas equation of state for the matter (provided that the density is not too high).
- A blackbody equation of state for the radiation.

In that case we may write for the pressure $P$ and internal energy $U$

$$P = \frac{N}{V} kT + \frac{aT^4}{3}$$

$$U = uV = C_v T + aT^4 V,$$

where the first term in each case is the contribution of the ideal gas and the second term is that of the radiation.
For mixtures of gas and radiation (common in high-temperature stellar environments) it is convenient to define a parameter $\beta$ that measures the relative contributions of gas pressure $P_g$ and radiation pressure $P_{\text{rad}}$ to the total pressure $P$:

$$\beta = \frac{P_g}{P}, \quad 1 - \beta = \frac{P_{\text{rad}}}{P}, \quad P = P_g + P_{\text{rad}}.$$

Therefore

- $\beta = 1$ implies that all pressure is generated by the gas.
- $\beta = 0$ implies that all pressure is generated by radiation.
- All values in between correspond to situations where pressure receives contributions from both gas and radiation.
Example: Assuming an ideal gas equation of state, the pressure generated by the gas alone in a mixture of ideal gas and radiation is

$$P_g = nkT = \beta P.$$  

Solving this equation for the total pressure,

$$P = \frac{nkT}{\beta} = \frac{\rho kT}{\beta \mu} = \frac{NkT}{\beta V},$$

which is of ideal gas form. Thus, the formal effect of mixing the radiation with the gas

- Produces an ideal gas equation of state
- But the gas has an effective mean molecular weight $\beta \mu$, where $\mu$ is the mean molecular weight for the gas alone.

Thus mixtures of ideal gases and radiation may be treated as modified ideal gases, but normally the relative contribution of radiation and gas to the pressure varies through the volume of a star so $\beta$ in realistic cases is a local function of position.
3.9.2 Adiabatic Systems of Gas and Radiation

The preceding discussion of gas and radiation mixtures depends only on the ideal gas assumption.

- Suppose we further confine discussion to adiabatic processes.

- From the adiabatic condition $\delta Q = 0$, the first law of thermodynamics, and the definition of $\beta$ we may show that at constant entropy (see Exercise 3.16)

$$\frac{d\ln T}{d\ln V} = \frac{-(\gamma - 1)(4 - 3\beta)}{\beta + 12(\gamma - 1)(1 - \beta)}.$$

- These logarithmic derivatives may then be used to evaluate the adiabatic exponents with the results

$$\Gamma_1 = \frac{d\ln P}{d\ln \rho} = \beta + \frac{(4 - 3\beta)^2(\gamma - 1)}{\beta + 12(1 - \beta)(\gamma - 1)}$$

$$\Gamma_2 = \left(1 - \frac{d\ln T}{d\ln P}\right)^{-1} = 1 + \frac{(4 - 3\beta)(\gamma - 1)}{\beta^2 + 3(\gamma - 1)(1 - \beta)(4 + \beta)}$$

$$\Gamma_3 = 1 + \frac{d\ln T}{d\ln \rho} = 1 + \frac{(4 - 3\beta)(\gamma - 1)}{\beta + 12(1 - \beta)(\gamma - 1)}$$
The adiabatic exponents $\Gamma_1$, $\Gamma_2$, and $\Gamma_3$ are plotted in Fig. 3.3 as a function of the parameter $\beta$ governing the relative contribution of gas and radiation to the pressure. They have the expected limiting behavior:

- Assuming $\gamma = \frac{5}{3}$ for a monatomic ideal gas and $\beta = 1$ (no radiation contribution to pressure) gives
  
  $\Gamma_1 = \Gamma_2 = \Gamma_3 = \frac{5}{3},$

- For $\beta = 0$ (all pressure generated by radiation) we find that the adiabatic exponents $\Gamma_1$, $\Gamma_2$, and $\Gamma_3$ all converge to a value of $\frac{4}{3}$.

- For other values of $\beta$ the adiabatic exponents are generally not equal and lie between $\frac{4}{3}$ and $\frac{5}{3}$.