The term covariance implies a formalism in which the laws of physics maintain the same form under a specified set of transformations.

EXAMPLE: Lorentz covariance implies equations that are constructed in such a way that they do not change their form under Lorentz transformations (three boosts between inertial systems and three rotations).^a

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^aAn inertial system is a frame of reference in which Newton’s first law of motion holds. Thus, for example, rotating frames and accelerated frames are not inertial. An inertial system is therefore in uniform translational motion with respect to any other inertial frame.
3.1 Covariance and Tensor Notation

- We shall be concerned generically with a transformation between one set of spacetime coordinates, denoted by

\[ x \equiv x^\mu = (x^0, x^1, x^2, x^3) \]

and a new set

\[ x'^\mu = x'^\mu(x) \quad \mu = 0, 1, 2, 3 \]

where \( x = x^\mu \) denotes the original (untransformed) coordinates.

This notation is an economical form of

\[ x'^\mu = \xi^\mu(x^1, x^2, x^3, x^4) \quad (\mu = 1, 2, \ldots) \]

where the single-valued, continuously differentiable functions \( \xi^\mu \) assign a new (primed) coordinate \( (x'^1, x'^2, x'^3, x'^4) \) to a point of the manifold with old coordinates \( (x^1, x^2, x^3, x^4) \). This transformation may be abbreviated to \( x'^\mu = \xi^\mu(x) \) and, even more tersely, to \( x'^\mu = x'^\mu(x) \).

- Coordinates are just labels, so laws of physics cannot depend on them. This implies that the system \( x'^\mu \) is not privileged and therefore this transformation should be invertible.

- Notice carefully that we are talking about the same point described in two different coordinate systems.
As a minimum, we must consider the transformations of

- Fields
- Derivatives of fields
- Integrals of fields.

The first two are necessary to formulate equations of motion, and the latter enter into various conservation laws.

To facilitate this, we shall introduce a set of mathematical quantities called *tensors* that are a generalization of the idea of scalars and vectors to more components.
3.1.1 Scalar Transformation Law

Simplest possibility: a field has a single component (magnitude) at each point that is unchanged by the transformation

$$\varphi'(x') = \varphi(x).$$

Quantities such as $\varphi(x)$ that are unchanged under the coordinate transformation are called *scalars*.

**EXAMPLE:** Value of the temperature at different points on the surface of the Earth.
3.1.2 Vectors

The gradient of a scalar field $\varphi(x)$ obeys

$$\frac{\partial \varphi(x)}{\partial x'^\mu} = \sum \frac{\partial \varphi(x)}{\partial x^\nu} \frac{\partial x^\nu}{\partial x'^\mu} \equiv \frac{\partial \varphi(x)}{\partial x^\nu} \frac{\partial x^\nu}{\partial x'^\mu}$$

(All partial derivatives understood to be evaluated at same point $P$.)

**Einstein summation convention:**

- An index that is repeated, once as a superscript and once as a subscript, implies a summation over that index.
- Such an index is a dummy index that is removed by the summation and should not appear on the other side of the equation.
- A repeated (dummy) index may be replaced by any other index not already in use without altering equation: $A_\alpha B^\alpha = A_\beta B^\beta$.
- A superscript (subscript) in a denominator counts as a subscript (superscript) in a numerator.
- Greek indices ($\alpha, \beta, \ldots$) denote the full set of spacetime indices running over $0, 1, 2, 3$.
- Roman indices ($i, j, \ldots$) denote the indices $1, 2, 3$ running only over the spatial coordinates.
- Placement of indices matters: generally $x^\alpha$ and $x_\alpha$ will be different quantities.
- At all stages of manipulating equations, the indices on the two sides of an equation (including their up or down placement) must match.
We can classify tensors according to a notation $t^m_n$, where $n$ is the number of lower indices and $m$ is the number of upper indices.

- Thus a scalar is a tensor of type $t^0_0$, since it carries no indices.
- The sum of $n$ and $m$ is the rank of the tensor. A scalar is a tensor of rank zero.

There are two kinds of rank-1 tensors, having the index pattern $t^0_1$ and $t^1_0$, respectively. The first is called a *covariant vector*:

**COVARIANT VECTOR:** A tensor having a transformation law that mimics that of the scalar field gradient,

$$A'_\mu(x') = \frac{\partial x^\nu}{\partial x'^\mu} A_\nu(x)$$

(covariant vector)

is of type $t^0_1$ and is termed a *covariant vector* or *1-form*.

**ECONOMY OF NOTATION:** The preceding equation really means four equations:

$$A'_\mu = \frac{\partial x^0}{\partial x'^\mu} A_0 + \frac{\partial x^1}{\partial x'^\mu} A_1 + \frac{\partial x^2}{\partial x'^\mu} A_2 + \frac{\partial x^3}{\partial x'^\mu} A_3 \quad (\mu = 0, 1, 2, 3)$$

each containing four terms. It is equivalent to the matrix equation

$$\begin{pmatrix}
A'_0 \\
A'_1 \\
A'_2 \\
A'_3
\end{pmatrix} = \begin{pmatrix}
\frac{\partial x^0}{\partial x^0} & \frac{\partial x^1}{\partial x^0} & \frac{\partial x^2}{\partial x^0} & \frac{\partial x^3}{\partial x^0} \\
\frac{\partial x^0}{\partial x^1} & \frac{\partial x^1}{\partial x^1} & \frac{\partial x^2}{\partial x^1} & \frac{\partial x^3}{\partial x^1} \\
\frac{\partial x^0}{\partial x^2} & \frac{\partial x^1}{\partial x^2} & \frac{\partial x^2}{\partial x^2} & \frac{\partial x^3}{\partial x^2} \\
\frac{\partial x^0}{\partial x^3} & \frac{\partial x^1}{\partial x^3} & \frac{\partial x^2}{\partial x^3} & \frac{\partial x^3}{\partial x^3}
\end{pmatrix} \begin{pmatrix}
A_0 \\
A_1 \\
A_2 \\
A_3
\end{pmatrix}.$$
CONTRAVARIANT VECTORS: A differential transforms like

\[ dx'^\mu = \frac{\partial x'^\mu}{\partial x^v} dx^v, \]

which suggests a second rank-1 transformation rule

\[ A'^\mu (x') = \frac{\partial x'^\mu}{\partial x^v} A^v (x) \quad \text{(contravariant vector)}. \]

A tensor that behaves in this way is of type \( t^1_0 \) and is termed a contravariant vector.
Therefore, we expect the possibility of two rank-1 tensors:

1. **Covariant vectors (1-forms)**, which carry a lower index and transform like the gradient of a scalar:
   \[ A'_\mu(x') = \frac{\partial x^\nu}{\partial x'^\mu} A_\nu(x) \]  
   (covariant vector)

2. **Contravariant vectors**, which carry an upper index and transform like the coordinate differential:
   \[ A'^\mu(x') = \frac{\partial x^\nu}{\partial x'^\mu} A_\nu(x) \]  
   (contravariant vector).

In the general case they must be distinguished (by placement—upper or lower—of their indices).
3.1.3 Scalar Product

Covariant and contravariant vectors permit defining a scalar product

\[ A \cdot B \equiv A_\mu B^\mu = A^\mu B_\mu \]

This transforms as a scalar because from

\[ A'_\mu (x') = \frac{\partial x^v}{\partial x'^\mu} A_v(x) \quad A'^\mu (x') = \frac{\partial x'^\mu}{\partial x^v} A_v(x). \]

we have that

\[ A' \cdot B' = A'_\mu B'^\mu = \frac{\partial x^v}{\partial x'^\mu} A_v \frac{\partial x'^\mu}{\partial x^\alpha} B^\alpha = \frac{\partial x^v}{\partial x^\alpha} \frac{\partial x'^\mu}{\partial x^\alpha} A_v B^\alpha \]

\[ = \frac{\partial x^v}{\partial x^\alpha} A_v B^\alpha = \delta_\alpha^v A_v B^\alpha \]

\[ = A_\alpha B^\alpha = A \cdot B, \]

where we have introduced the \textit{Kronecker delta} through

\[ \delta_\mu^v = \frac{\partial x'^\mu}{\partial x^v} = \frac{\partial x^\mu}{\partial x^v} = \begin{cases} 1 & \mu = v \\ 0 & \mu \neq v \end{cases} \]

Eliminating indices by summing over them in tensor products is called \textit{contraction}. The scalar product has no tensor indices left and is said to be \textit{fully contracted}. 

3.1.4 Rank-2 Tensors

We may distinguish three kinds of rank-2 tensors according to the transformation laws

\[ T'_{\mu\nu} = \frac{\partial x^\alpha}{\partial x'^\mu} \frac{\partial x^\beta}{\partial x'^\nu} T_{\alpha\beta} \]

\[ T'^{\nu}_{\mu} = \frac{\partial x^\alpha}{\partial x'^\mu} \frac{\partial x'^\nu}{\partial x^\beta} T_{\alpha}^{\beta} \]

\[ T'^{\mu\nu} = \frac{\partial x'^\mu}{\partial x^\alpha} \frac{\partial x'^\nu}{\partial x^\beta} T_{\alpha\beta} \]

This may easily be generalized to tensors of any rank.

- **Covariant Tensors**: carry only lower indices
- **Contravariant Tensors**: carry only upper indices
- **Mixed Tensors**: carry both upper and lower indices

**EXAMPLE**: the Kronecker delta $\delta^\nu_{\mu}$ is a mixed tensor of rank 2.

Memory aid:

- Each upper index $\mu$ on left side requires right-side “factor” of form $\partial x'^\mu / \partial x^\nu$ (prime in numerator).
- Each lower index $\nu$ on left side requires right-side “factor” of form $\partial x^\mu / \partial x'^\nu$ (prime in denominator).
- “Position of index = position of primed coordinate”
NOTE: Not all quantities carrying indices are tensors; it is the transformation laws that define the tensors.

NOTE: We employ a standard shorthand by using “a tensor $T_{\mu\nu}$” to mean “a tensor with components $T_{\mu\nu}$.”
3.1.5 Metric Tensor

A rank-2 tensor of particular importance is the metric tensor $g_{\mu\nu}$ because it is associated with the line element

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu.$$ 

It is symmetric ($g_{\mu\nu} = g_{\nu\mu}$) and satisfies the usual rank-2 tensor transformation law

$$g'_{\mu\nu} = \frac{\partial x^\alpha}{\partial x'^\mu} \frac{\partial x^\beta}{\partial x'^\nu} g_{\alpha\beta}.$$ 

The contravariant metric tensor $g^{\mu\nu}$ is defined by the requirement

$$g_{\mu\alpha}g^{\alpha\nu} = \delta^\nu_\mu.$$ 

(Thus, $g_{\mu\nu}$ and $g^{\mu\nu}$ are matrix inverses.)

We may use contraction with the metric tensor to raise and lower tensor indices; for example

$$A^\mu = g^{\mu\nu}A_\nu \quad A_\mu = g_{\mu\nu}A^\nu.$$ 

Thus, the scalar product of vectors may also be expressed as

$$A \cdot B = g_{\mu\nu}A^\mu B^\nu \equiv A_\nu B^\nu.$$
3.1.6 Antisymmetric 4th-Rank Tensor

A rank-4 tensor $\varepsilon_{\alpha\beta\gamma\delta}$ of particular importance may be introduced by the requirement that

- $\varepsilon_{0123} = 1$

- $\varepsilon_{\alpha\beta\gamma\delta}$ be completely antisymmetric in the exchange of any two indices (thus it must vanish if any two indices are the same).

This tensor is commonly called the completely antisymmetric 4th-rank tensor or the Levi–Civita symbol.
3.2 Symmetric and Antisymmetric Tensors

The symmetry of tensors under exchanging pairs of indices is often important.

- An arbitrary rank-2 tensor can always be decomposed into a symmetric and antisymmetric part:

\[ T_{\alpha\beta} = \frac{1}{2} (T_{\alpha\beta} + T_{\beta\alpha}) + \frac{1}{2} (T_{\alpha\beta} - T_{\beta\alpha}), \]

where the first term is clearly symmetric and the second term antisymmetric under exchange of indices.

- For completely symmetric and completely antisymmetric rank-2 tensors we have

\[ T_{\alpha\beta} = \pm T_{\beta\alpha} \quad T^{\alpha\beta} = \pm T^{\beta\alpha} \quad T_{\alpha}^{\beta} = \pm T_{\beta}^{\alpha}, \]

where the plus sign holds if the tensor is symmetric and the minus sign if it is antisymmetric.

- More generally, we say that a tensor of rank two or higher is symmetric in any two of its indices if exchanging those indices leaves the tensor invariant and antisymmetric (sometimes termed skew-symmetric) in any two indices if it changes sign upon switching those indices.
• It is common to denote symmetrizing and antisymmetrizing operations on tensor indices by a bracket notation in which ( ) indicates symmetrization and [ ] indicates antisymmetrization over indices included in the brackets.

• For example, symmetrization over all indices for a rank-\( N \) covariant tensor \( T_{\alpha, \beta, \ldots, \omega} \) corresponds to

\[
T(\alpha, \beta, \ldots, \omega) \equiv \frac{1}{N!} \text{(Sum over permutations on indices } \alpha, \beta, \ldots \omega) \]

and antisymmetrization over all indices of \( T_{\alpha, \beta, \ldots, \omega} \) corresponds to

\[
T[\alpha, \beta, \ldots, \omega] \equiv \frac{1}{N!} (\pm \text{Sum over permutations on indices } \alpha, \beta, \ldots \omega),
\]

where the notation \( \pm \) indicates that the terms of the sum have a plus sign if they correspond to an even number of index exchanges and a negative sign if they correspond to an odd number of index exchanges.

• Thus, for rank-2 contravariant tensors we may write

\[
T^{(\alpha\beta)} = \frac{1}{2}(T^{\alpha\beta} + T^{\beta\alpha}) \quad T^{[\alpha\beta]} = \frac{1}{2}(T^{\alpha\beta} - T^{\beta\alpha})
\]
• In the more general case one may be interested in symmetrizing or antisymmetrizing over only a subset of indices for higher-rank tensors.

• If the indices are contiguous the above notation suffices with only the indices to be symmetrized or antisymmetrized included in the brackets.

• In the event that indices to be symmetrized or antisymmetrized are not adjacent to each other, the preceding notation may be extended by using vertical brackets to exclude indices from the symmetrization or antisymmetrization.

**Example:** The expression

\[ T_{\alpha|\beta|\gamma|\delta|} = \frac{1}{2} (T_{\alpha\beta\gamma\delta} - T_{\alpha\delta\gamma\beta}) \]

corresponds to a rank-4 covariant tensor that has been antisymmetrized in its second and fourth indices only.
3.2. SYMMETRIC AND ANTISYMMETRIC TENSORS

3.2.1 Invariant Integration

Change of volume elements for spacetime integration:

\[ d^4x = \det \left( \frac{\partial x}{\partial x'} \right) d^4x' \]

where \( \det(\partial (x)/\partial (x')) \) is the Jacobian determinant of the transformation between the coordinates.

Example: For \((x,y) \leftrightarrow (u,v)\) the Jacobian is the determinant of

\[ \begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{pmatrix} \]

or its inverse.

The metric tensor transforms as

\[ g'_{\mu\nu} = \frac{\partial x^\alpha}{\partial x'^\mu} \frac{\partial x^\beta}{\partial x'^\nu} g_{\alpha\beta} \quad \text{(Triple matrix product)} \]

Therefore, since:

\text{determinant of a product} = \text{product of determinants}

the determinant of the metric tensor \( g \equiv \det g_{\mu\nu} \) transforms as

\[ g' = \det \left( \frac{\partial x}{\partial x'} \right) \det \left( \frac{\partial x}{\partial x'} \right) g \quad \rightarrow \quad \det \left( \frac{\partial x}{\partial x'} \right) = \frac{\sqrt{|g'|}}{\sqrt{|g|}} \]

which gives when inserted into the first equation

\[ \sqrt{|g|} d^4x = \sqrt{|g'|} d^4x' \]

\(|g| \) because \( g \) can be negative in 4-D spacetime).
This is a scalar expression and thus its value is independent of coordinates. It follows that an integral of the form

\[ I = \int \varphi(x) \sqrt{|g|} d^4x \]

with fixed boundaries is a scalar if \( \varphi(x) \) is a scalar.

In integrals we shall employ

\[ dV = \sqrt{|g|} d^4x \]

as an invariant volume element.
Example: The metric for a 2-dimensional spherical surface (2-sphere) is specified by the line element

$$ds^2 = g_{ij}dx^i dx^j,$$

which is explicitly in spherical coordinates

$$d\ell^2 = R^2d\theta^2 + R^2\sin^2\theta d\phi^2.$$

This may be written as the matrix equation

$$d\ell^2 = (d\theta \quad d\phi) \begin{pmatrix} R^2 & 0 \\ 0 & R^2\sin^2\theta \end{pmatrix} \begin{pmatrix} d\theta \\ d\phi \end{pmatrix}.$$

The area of the 2-sphere may then be expressed as the “invariant volume integration”

$$A = \int \sqrt{|g|} d^2x = \int_0^{2\pi} d\varphi \int_0^\pi \sqrt{\det g_{ij}} d\theta$$

$$= \int_0^{2\pi} d\varphi \int_0^\pi R^2 \sin \theta d\theta = 4\pi R^2.$$

where the metric tensor $g_{ij}$ is the $2 \times 2$ matrix in the preceding equation for the line element.

In this 2-dimensional example the sign of the determinant is positive, so no absolute value is required under the radical.
3.2.2 Covariant Derivatives

Let us now consider the derivatives of tensor quantities. First introduce two common compact notations for partial derivatives

\[
\varphi, \mu \equiv \frac{\partial \varphi(x)}{\partial x^\mu} \quad \partial_\mu \varphi \equiv \frac{\partial \varphi(x)}{\partial x^\mu}
\]

The derivative of a scalar is a covariant vector and scalars and their derivatives are well-defined tensors. But, for the derivative of a covariant vector, by using the rule for the derivative of a product

\[
A'_\mu, \nu \equiv \frac{\partial A'_\mu}{\partial x'^\nu} = \frac{\partial}{\partial x'^\nu} \left( A_\alpha \frac{\partial x^\alpha}{\partial x'^\mu} \right)_{A'_\mu} \\
= \frac{\partial A_\alpha}{\partial x'^\nu} \frac{\partial x^\alpha}{\partial x'^\mu} + A_\alpha \frac{\partial^2 x^\alpha}{\partial x'^\nu \partial x'^\mu}
\]

\[
= \frac{\partial A_\alpha}{\partial x'^\nu} \frac{\partial x^\beta}{\partial x'^\mu} \frac{\partial x^\alpha}{\partial x'^\nu} \frac{\partial x^\alpha}{\partial x'^\mu} + A_\alpha \frac{\partial^2 x^\alpha}{\partial x'^\nu \partial x'^\mu}
\]

\[
= A_{\alpha, \beta} \frac{\partial x^\beta}{\partial x'^\nu} \frac{\partial x^\alpha}{\partial x'^\mu} + A_\alpha \frac{\partial^2 x^\alpha}{\partial x'^\nu \partial x'^\mu}
\]

In curved spacetime it is not possible to transform away the second term globally: *Partial differentiation of tensors is NOT a covariant operation in curved spacetime (for tensors of rank 1 or higher).*

But if we introduce the Christoffel symbols $\Gamma^\lambda_{\alpha\beta}$ and require that they obey (not a tensor—see the 2nd derivatives!)

$$\Gamma^\prime{}_{\alpha\beta} = \Gamma^\kappa_{\mu\nu} \frac{\partial x^\mu}{\partial x^\prime{}^\alpha} \frac{\partial x^\nu}{\partial x^\prime{}^\beta} \frac{\partial x^\lambda}{\partial x^\kappa} + \frac{\partial^2 x^\mu}{\partial x^\prime{}^\alpha} \frac{\partial x^\prime{}^\beta}{\partial x^\mu} \frac{\partial x^\lambda}{\partial x^\kappa},$$

we may show that (Exercise),

$$\left( A^\prime_{\mu\nu} - \Gamma^\lambda_{\mu\nu} A^\prime_{\lambda} \right) = \left( A_{\alpha\beta} - \Gamma^\kappa_{\alpha\beta} A_\kappa \right) \frac{\partial x^\alpha}{\partial x^\mu} \frac{\partial x^\beta}{\partial x^\nu}.\]$$

For the quantity in parentheses this is the transformation law for a rank-2 covariant tensor:

$$B^\prime_{\mu\nu} = B_{\alpha\beta} \frac{\partial x^\alpha}{\partial x^\mu} \frac{\partial x^\beta}{\partial x^\nu}.$$

This suggests that we define the covariant derivative of a vector as

$$A_{\mu;\nu} \equiv \underbrace{A_{\mu,\nu}}_{\text{tensor}} - \underbrace{\Gamma^\lambda_{\mu\nu} A_\lambda}_{\text{not tensor}}$$

where a subscript comma denotes ordinary partial differentiation and a subscript semicolon denotes covariant differentiation with respect to the variables following it.

(We shall also employ the notation $D_\mu$ to denote an operator that takes the covariant derivative with respect to $x^\mu$.)

The covariant derivative of a covariant vector then transforms as a covariant tensor of rank 2, even though neither of its terms is a tensor.
Likewise, we can introduce the covariant derivative of a contravariant vector

\[ A_{\lambda}^{\mu} = A_{\lambda}^{\mu} + \Gamma_{\alpha \mu}^{\lambda} A^{\alpha}, \]

and the covariant derivatives of the three possible rank-2 tensors through

\[ A_{\mu \nu; \lambda} = A_{\mu \nu, \lambda} - \Gamma_{\mu \lambda}^{\alpha} A_{\alpha \nu} - \Gamma_{\nu \lambda}^{\alpha} A_{\mu \alpha} \]

\[ A_{\nu; \kappa}^{\mu} = A_{\nu, \kappa}^{\mu} + \Gamma_{\alpha \kappa}^{\mu} A_{\alpha \nu} - \Gamma_{\nu \kappa}^{\alpha} A_{\mu \alpha} \]

\[ A_{\nu}^{\mu \nu; \kappa} = A_{\nu}^{\mu \nu, \kappa} + \Gamma_{\alpha \kappa}^{\mu} A_{\alpha \nu} + \Gamma_{\nu \kappa}^{\alpha} A_{\mu \alpha} \]

(which are rank-3 tensors), and so on.

Heuristic rule for constructing the covariant derivative of a tensor having any rank:

- Form the ordinary partial derivative of the tensor
- Add one Christoffel symbol term having the sign and form for a covariant vector for each lower index of the tensor
- Add one Christoffel symbol term having the sign and form for a contravariant vector for each upper index of the tensor
Most rules for partial differentiation carry over with suitable generalization for covariant differentiation.

**Example:** Covariant derivative of product

\[(A_\mu B_\nu)_{;\lambda} = A_{\mu;\lambda} B_\nu + A_\mu B_{\nu;\lambda}.\]

which is the usual (Leibniz rule) result.

The most important exception concerns the properties of successive covariant differentiations. Although partial derivative operators normally commute, covariant derivative operators generally do not commute with each other.

One important consequence of covariant differentiation (Exercise):

\[D_\alpha g_{\mu\nu} = g_{\mu\nu;\alpha} = 0,\]

Some implications:

- Raising and lowering index by contraction with \(g_{\mu\nu}\) commutes with covariant differentiation.

- This will allow in the Einstein field equations a *vacuum energy term* (accelerated expansion and dark energy).
3.2.3 Invariant Equations

The properties of tensors elaborated above ensure that any equation will be invariant under general coordinate transformations provided that it equates tensors having the same upper and lower indices.

EXAMPLES:

- If the quantities $T_{\mu}^{\nu}$ and $U_{\mu}^{\nu}$ each transform as mixed rank-2 tensors and $T_{\mu}^{\nu} = U_{\mu}^{\nu}$ in the $x$ coordinate system, then in the $x'$ coordinate system $T'_{\mu}^{\nu} = U'_{\mu}^{\nu}$.

- An equation that equates any tensor to zero is invariant under general coordinate transformations.

- Equations such as $T_{\mu}^{\nu} = 10$ or $T^{\mu} = U_{\mu}$ generally are not valid in all coordinate systems because they equate tensors of different kinds.