A fundamental property of main sequence stars like our Sun is their stability over long periods of time.

- The geological record indicates that the Sun has been emitting energy at its present rate for several billion years, with relatively small variation.
- The key to this stability is that main sequence stars are in a state of near perfect hydrostatic equilibrium.
- In hydrostatic equilibrium the pressure gradients produced by thermonuclear fusion and internal heat almost exactly balance the gravitational forces.

Thus the starting point for an understanding of stellar structure is an understanding of hydrostatic equilibrium and departures from that equilibrium.
4.1 Newtonian Gravitation

The Newtonian gravitational field is derived from a gravitational potential $\Phi$ that obeys the Poisson equation, which for spherical symmetry is

$$\frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 \frac{\partial \Phi}{\partial r} \right) = 4\pi G \rho.$$ 

The gravitational acceleration is given by

$$g = \frac{\partial \Phi}{\partial r} = \frac{Gm}{r^2},$$

where $m = m(r)$ is the mass contained within the radius $r$. Hence, for spherical geometry

$$\Phi(r) = \int_0^r g \, dr + \text{constant} = \int_0^r \frac{Gm(r)}{r^2} \, dr + \text{constant}.$$ 

The constant is fixed by requiring that $\Phi \to 0$ as $r \to \infty$. 
4.2 Conditions for Hydrostatic Equilibrium

The local gravitational acceleration at a radius $r$ is given by

$$g = \frac{\partial \Phi}{\partial r} = \frac{Gm}{r^2},$$

where $m(r)$ is the mass contained within a radius $r$. The mass contained in a thin spherical shell is (see Fig. 4.1)

$$dm = m(r + dr) - m(r) = 4\pi r^2 \rho(r) dr.$$

Integrating this from the origin to a radius $r$ yields the **mass function** $m(r)$,

$$m(r) = \int_0^r 4\pi r^2 \rho \, dr.$$
Now consider the total gravitational force acting on a volume of unit area in the concentric sphere of radius \( r \) and depth \( dr \).

- The magnitude of this force (per unit area) will be

\[
F_g = -\rho g(r) dr = -\rho \frac{Gm(r)}{r^2} dr,
\]

Negative sign \( \rightarrow \) directed toward the center of the sphere.

- The force per unit area resulting from the pressure difference between \( r \) and \( r + dr \) is

\[
P(r) - P(r + dr) = -\frac{\partial P}{\partial r} dr
\]

Negative sign \( \rightarrow \) directed outward.

- The inwardly directed gravitational force is counterbalanced by a net outward force arising from the pressure gradient of the gas and radiation that has a magnitude

\[
F_p = P(r) - P(r + dr) = -\frac{\partial P}{\partial r} dr.
\]
4.2. CONDITIONS FOR HYDROSTATIC EQUILIBRIUM

- The total force acting on this volume of unit surface area is then

\[ F = F_g + F_p = -\frac{\partial P}{\partial r} dr - \frac{Gm(r)}{r^2} \rho dr, \]

- by Newton’s 2nd law the equation of motion is

\[ F = ma = \rho dr \frac{\partial^2 r}{\partial t^2}, \]

- This leads to

\[ \rho \frac{\partial^2 r}{\partial t^2} = -\frac{\partial P}{\partial r} - \frac{Gm(r)}{r^2} \rho. \]

- For hydrostatic equilibrium, the left side vanishes because the acceleration \( \frac{\partial^2 r}{\partial t^2} = 0 \) and we obtain

\[ \frac{dP}{dr} = -\frac{Gm(r)}{r^2 g} \rho = -g \rho, \]

where partial derivatives have been replaced with derivatives because by our assumption there is no longer any time dependence.
Hydrostatic Equilibrium and Stellar Interiors: In the equation
\[ \frac{dP}{dr} = -\frac{Gm(r)}{r^2}\rho = -g\rho, \]
both \( \rho \) and \( \frac{Gm(r)}{r^2} \) are positive.

1. Thus \( \frac{dP}{dr} \leq 0 \) and pressure must decrease outward everywhere for a gravitating system to be in hydrostatic equilibrium.

   \[ \frac{dP}{dr} \text{ is always negative under conditions of hydrostatic equilibrium.} \]

2. This will in turn imply that density and temperature must increase toward the center of a star.

   Thus, the conditions of hydrostatic equilibrium are sufficient to ensure that stars must be much more dense and hot near their centers than near their surfaces.
4.2. CONDITIONS FOR HYDROSTATIC EQUILIBRIUM

The equations

\[
\frac{dP}{dr} = -\frac{Gm(r)}{r^2}\rho = -g\rho,
\]

\[
dm = 4\pi r^2 \rho(r)dr.
\]

are our first two equations of stellar structure.

• They constitute two equations in three unknowns \((P, m, \text{ and } \rho)\) as functions of \(r\).

• This system of equations may be \textit{closed by specifying an equation of state} relating these quantities.

Before considering that, we explore some consequences that follow from these equations alone.
4.3 Lagrangian and Eulerian Descriptions

In studying fluid motion, there are two basic computational points of view that we can take.

1. We can fix a grid and watch the fluid flow through the grid; this is called *Eulerian hydrodynamics*.

2. Alternatively, we can construct coordinates that are attached to the mass elements and move with them; this is called *Lagrangian hydrodynamics*.

To appreciate the difference, consider determining the temperature of the atmosphere over time either from a weather balloon drifting with the wind or from a fixed point on the ground.

- The first is a Lagrangian point of view, if we imagine the balloon to be tied approximately to the motion of a fixed packet of air.
- The second is Eulerian, since one observes the air from a fixed observation point as it flows by.

3. In the limit that accelerations of the fluid can be neglected, Lagrangian and Eulerian descriptions of hydrodynamics reduce to Lagrangian and Eulerian descriptions of hydrostatics.
4.3.1 Lagrangian Formulation of Hydrostatics

Let us illustrate the Lagrangian approach by reformulating the preceding equations with \( m(r) \) rather than \( r \) as the independent variable.

- The general result for a change of variables between Eulerian and Lagrangian representations, \( (r, t) \rightarrow (m, t) \), is

\[
\frac{\partial}{\partial m} = \frac{\partial}{\partial r} \cdot \frac{\partial r}{\partial m} \quad \left( \frac{\partial}{\partial t} \right)_m = \frac{\partial}{\partial r} \left( \frac{\partial r}{\partial t} \right)_m + \left( \frac{\partial}{\partial t} \right)_r.
\]

where the subscripts denote variables held constant.

- Apply the first of these to the mass parameter \( m \) and use \( dm = 4\pi r^2 \rho(r) dr \) to obtain

\[
\frac{dr}{dm} = \frac{1}{4\pi r^2 \rho},
\]

In operator form, the transformation between the two representations is

\[
\frac{\partial}{\partial m} = \frac{1}{4\pi r^2 \rho} \frac{\partial}{\partial r}.
\]

- Now we use this to convert

\[
\frac{dP}{dr} = -\frac{Gm(r)}{r^2}\rho
\]

to Lagrangian coordinates, giving

\[
\frac{dP}{dm} = -\frac{Gm}{4\pi r^4} \quad \text{(Lagrangian form)}.
\]
Table 4.1: Equations of hydrostatics

<table>
<thead>
<tr>
<th>Eulerian coordinates ((r, t))</th>
<th>Lagrangian coordinates ((m, t))</th>
</tr>
</thead>
<tbody>
<tr>
<td>(\frac{dm}{dr} = 4\pi r^2 \rho)</td>
<td>(\frac{dr}{dm} = \frac{1}{4\pi r^2 \rho})</td>
</tr>
<tr>
<td>(\frac{dP}{dr} = -\frac{Gm \rho}{r^2})</td>
<td>(\frac{dP}{dm} = -\frac{Gm}{4\pi r^4})</td>
</tr>
</tbody>
</table>

Table 4.1 summarizes the equations of spherical hydrostatics in Eulerian and Lagrangian form.
4.3.2 Contrasting Lagrangian and Eulerian Descriptions

Eulerian and Lagrangian representations have advantages and disadvantages in a particular context.

• Our observational mindset is often Eulerian:

  We tend to think of monitoring a river by placing a measuring device at a fixed point on the river rather than imagining a measuring device floating down the river with a given packet of water.

• We tend to formulate laws of physics in a Lagrangian way:

  For collision of billiard balls, we normally imagine following each ball. We seldom imagine staking out points on the table and asking how balls move past those fixed points (a clearly Eulerian point of view).
Because the Lagrangian point of view is often more simply tied to the underlying physical laws, the Lagrangian formulation is often preferred when there are clear symmetries and conservation laws.

Example: Imagine a spherical star that is neither gaining nor losing mass, but is pulsating radially in size.

- The radial distance to the surface (Eulerian coordinate) is changing with time.
- Mass contained within the outermost radius (Lagrangian coordinate) is constant in time.

On the other hand, if spherical symmetry is broken and there is convective and turbulent motion of the fluid, the Eulerian description is often simpler than the Lagrangian description.
A particularly important concept in astrophysics is that of a *dynamical timescale*.

- **Reason:** a dynamical timescale sets the order of magnitude for the time required for a system to respond to a perturbation.

- The dynamical response of stars to perturbations of their hydrostatic equilibrium is of obvious significance in understanding stars and their evolution.

- Consider the *free-fall timescale* $t_{\text{ff}}$

$$ t_{\text{ff}} \simeq \sqrt{\frac{1}{G\bar{\rho}}} \simeq \sqrt{\frac{R}{g}} $$

where

$$ \bar{\rho} = \frac{M}{\frac{4}{3}\pi R^3} $$

is the average density and

$$ g = \frac{GM}{R^2} $$

is the gravitational acceleration.

- This defines a timescale for gravitational collapse of a uniform-density sphere if it suddenly lost all pressure support.
• We may introduce a second dynamical timescale by considering the opposite extreme: if gravity were taken away, how fast would the star expand by virtue of its pressure?

• This timescale can depend only on $R$, $\bar{\rho}$, and $\bar{P}$, and the only combination of these quantities having time units is

$$t_{\text{exp}} \approx R \sqrt{\frac{\bar{\rho}}{\bar{P}}} \approx \frac{R}{\bar{v}_s},$$

This characteristic expansion timescale has a simple physical interpretation:

1. $(\rho/P)^{1/2}$ is approximately the inverse of the mean sound speed $\bar{v}_s$ for the medium.

2. This implies that $t_{\text{exp}}$ is approximately the time for a sound wave to travel from the center to the surface of the star.

This makes sense because pressure waves propagating outward should be characterized by that timescale.

• Hydrostatic equilibrium will clearly be precarious unless the two dynamical timescales are comparable with each other; therefore, we define a hydrodynamical timescale for the system through

$$\tau_{\text{hydro}} \approx t_{\text{exp}} \approx t_{\text{ff}} \approx \sqrt{\frac{1}{G\bar{\rho}}}.$$
### 4.4. DYNAMICAL TIMESCALES

#### Table 4.2: Hydrodynamical timescales

<table>
<thead>
<tr>
<th>Object</th>
<th>$\sim M/M_\odot$</th>
<th>$\sim R/R_\odot$</th>
<th>$\bar{\rho}/\rho_\odot$</th>
<th>$\tau_{\text{hydro}}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Red Giant</td>
<td>1</td>
<td>100</td>
<td>$10^{-6}$</td>
<td>36 days</td>
</tr>
<tr>
<td>Sun</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>55 minutes</td>
</tr>
<tr>
<td>White Dwarf</td>
<td>1</td>
<td>1/50</td>
<td>$10^5$</td>
<td>9 seconds</td>
</tr>
</tbody>
</table>

**Example:** For the Sun $\bar{\rho} = 1.4 \text{ g cm}^{-3}$ and

$$
\tau_{\text{hydro}} \approx \sqrt{\frac{1}{G\bar{\rho}}} \approx 55 \text{ minutes}.
$$

- If hydrostatic equilibrium were not satisfied we would expect to see changes in a matter of hours.
- But the fossil record indicates that the Sun has been extremely stable for billions of years.
- We conclude that the Sun is in very good hydrostatic equilibrium.

In Table 4.2 we illustrate the hydrodynamical timescale for several kinds of stars calculated using this formula.
Stars have at their disposal two large sources of energy:

1. \textit{Gravitational energy}, which can be released by contraction.

2. \textit{Internal energy}, which can be produced both by contraction and by fusion and other internal processes.

We now derive an important relationship between internal and gravitational energy for objects in approximate hydrostatic equilibrium called the \textit{virial theorem}.
We may multiply both sides of the Lagrangian equation
\[ \frac{dP}{dm} = -\frac{Gm}{4\pi r^4} \]
by \(4\pi r^3\) and integrate over \(dm\) from 0 to \(M \equiv m(R)\) to give

\[
\int_0^M \frac{Gm}{r} \, dm = -4\pi \int_0^M r^3 \frac{\partial P}{\partial m} \, dm \]

Integrate by parts

\[
= -4\pi r^3 P \bigg|_{m=0}^{m=M} + 12\pi \int_0^M r^2 P \frac{\partial r}{\partial m} \, dm
\]

identically zero

\[
= 12\pi \int_0^M r^2 P \frac{1}{4\pi r^2 \rho} \, dm
\]

\[
= \int_0^M \frac{3P}{\rho} \, dm,
\]

- \(\rho, r,\) and \(P\) are functions of independent variable \(m\)
- An integration by parts was used to obtain line 2
- In the first term of line 2
  1. \(r\) vanishes when \(m = 0\) (center of star)
  2. \(P\) vanishes when \(m = M\) (surface of star).

Thus this term is identically zero.

- \(\frac{dr}{dm} = \frac{1}{4\pi r^2 \rho}\) was used in going from line 2 to line 3
The equation just obtained,
\[ \int_0^M \frac{GMm}{r} \, dm = \int_0^M \frac{3P}{\rho} \, dm, \]
has a simple interpretation. First consider the right side:

- \( P/\rho = kT/\mu \) for an ideal monatomic gas
- Thus the right side is twice the internal energy \( U \) because
  \[ \int_0^M \frac{3P}{\rho} \, dm = \frac{3kT}{\mu} \int_0^M \frac{M}{\mu} \, dm = \frac{3MkT}{\mu} \]
  \[ = 3\left( \frac{M}{\mu} \right) kT = 3NkT = 2U, \]
  since for an ideal monatomic gas \( U = \frac{3}{2}NkT \).

Hence we have
\[ \int_0^M \frac{GMm}{r} \, dm = \int_0^M \frac{3P}{\rho} \, dm \rightarrow \int_0^M \frac{GMm}{r} \, dm = 2U. \]

The integral on the left side may be interpreted by asking the question

What is the total gravitational energy released in forming a star?
Consider Fig. 4.2, where a shell of mass $\Delta m$ falls from infinity onto the surface of a spherical mass of radius $r$ and enclosed mass $m(r)$. The gravitational energy released is

$$d\Omega = \int_{\infty}^{r} F_g ds = \int_{\infty}^{r} g(s) \Delta m \, ds$$

$$= \int_{\infty}^{r} \frac{Gm(r)}{s^2} \frac{4\pi r^2 \rho \, dr \, ds}{\Delta m}$$

$$= -\frac{Gm(r)}{s} \bigg|_{\infty}^{r} \times 4\pi r^2 \rho \, dr = -4\pi r^2 \rho \, dr \, \frac{Gm(r)}{r},$$

and the total energy released in assembling a star of radius $R$ and mass $M$ from such mass shells is

$$\Omega = \int d\Omega = -4\pi \int_{0}^{R} r^2 \rho \frac{Gm(r)}{r} \, dr = -\int_{0}^{M} \frac{Gm(r)}{r} \, dm,$$

where $dm/dr = 4\pi r^2 \rho$ was used and $M \equiv m(R)$. 
Thus, for the left side of the virial theorem equation

\[ \int_0^M \frac{Gm(r)}{r} \, dm = -\Omega \]  
(Gravity potential energy of star),

and from the previous result for the right side of the virial theorem equation,

\[ \int_0^M \frac{Gm}{r} \, dm = 2U, \]

we see that for an ideal gas

\[ \int_0^M \frac{Gm}{r} \, dm = \int_0^M \frac{3P}{\rho} \, dm \quad \rightarrow \quad 2U + \Omega = 0 \]

This result is called the **Virial Theorem** (for a monatomic ideal gas):

\[ 2U + \Omega = 0, \]

where \( U \) is the internal energy of the star and \( \Omega \) is its gravitational energy.
The **virial theorem** for an ideal, monatomic gas,

\[2U + \Omega = 0 \quad \text{(or in the form } U = -\frac{1}{2} \Omega)\]

1. Establishes an important general relationship between the internal energy and gravitational energy of a star *in hydrostatic equilibrium*.

2. Is of broad applicability because
   
   - It was derived under very general conditions.
   - It relates the two most important energy reserves for a star, gravitational and internal energy.

We shall often use the virial theorem and concepts derived from it in discussions of stellar structure and evolution.
As a star forms, gravitational contraction releases an amount of energy $\Delta \Omega$.

- As stars form they go through a sequence of stages that are often nearly in hydrostatic equilibrium.

- Since the virial theorem must be satisfied for hydrostatic equilibrium to hold, as a newly-forming star contracts the thermal energy must change by

$$\Delta U \approx -\frac{1}{2} \Delta \Omega$$

and the excess energy must be radiated into space.

A cloud of gas and dust collapsing to form a star cannot be in hydrostatic equilibrium. However, through much of the collapse the nascent star is only slightly out of equilibrium, so we may expect the virial theorem to be approximately satisfied.

Thus, gravitational contraction has three consequences:

1. The star heats up,
2. Some energy is radiated into space,
3. The total energy of the star decreases and it becomes more bound.

This leads means that the star “heats up while it cools down”. 
How can a star “heat up while it cools down”?  
Answer: gravity.

The virial theorem is

\[ 2U = -\Omega. \]

Identifying \( U \) as the kinetic energy and \( \Omega \) as the potential energy, an alternative statement is

\[ E_{\text{kin}} = -\frac{1}{2}E_{\text{pot}}. \]

and if \( E = E_{\text{kin}} + E_{\text{pot}} \) is the total energy,

\[ E_{\text{kin}} = -E. \]

But this implies that

• Adding energy decreases the kinetic energy.

• Removing energy increases the kinetic energy.

Thus, identifying kinetic energy ↔ temperature,

Gravitating objects in hydrostatic equilibrium have negative heat capacities.

This is a general property of gravity because it is long-ranged and unscreened.
Because of quantum Hawking radiation, black holes have a temperature

\[ T = \frac{\hbar c^3}{8\pi kGM} \]

where \( M \) is the mass, \( k \) is Boltzmann’s constant, and \( G \) is the gravitational constant. As Hawking radiation is emitted

1. The black hole loses mass (energy), and
2. The temperature rises.

Like stars, a black hole (the ultimate gravitating system!) becomes hotter as it loses energy: it exhibits a negative heat capacity.
Returning to our contracting protostar, if approximate hydrostatic equilibrium is to be maintained,

1. At each infinitesimal step of the contraction the star must wait until half of the released gravitational energy is radiated before it can continue to contract.

2. This implies that there is a *timescale for contraction* in near hydrostatic equilibrium that is set by the time required to radiate the excess energy.

This contraction timescale is called the *Kelvin–Helmholtz timescale* for the system.
We may estimate the Kelvin–Helmholtz timescale roughly by assuming constant density $\rho$ and a corresponding mass contained within the radius $r$

$$m(r) = \frac{4}{3} \pi r^3 \rho$$

during the collapse. Then the gravitational energy released in collapsing down to a star of radius $R$ is

$$\Omega = - \int_0^R 4\pi r^2 \rho \frac{Gm}{r} \, dr \quad (\text{substitute } m = \frac{4}{3} \pi r^3 \rho)$$

$$= - \frac{16}{15} \pi^2 \rho^2 G R^5 \quad (\text{substitute } \rho = \frac{3M}{4\pi R^3})$$

$$= - \frac{3}{5} \frac{GM^2}{R},$$

where the total mass is

$$M = \frac{4}{3} \pi R^3 \rho$$

Taking $M = M_\odot$ and $R = R_\odot$, we find that $\Omega_\odot = 2.3 \times 10^{48}$ erg of gravitational energy was released in forming the Sun. By the virial theorem, half of this must be radiated while the Sun contracts:

$$E_{\text{rad}}^\odot = \frac{1}{2} \Omega_\odot \approx \frac{G M_\odot^2}{R_\odot} \approx 10^{48} \text{ erg.}$$

The Kelvin–Helmholtz timescale $t_{KH}$ is the characteristic time to radiate this energy.
We may make a rough estimate of the Kelvin–Helmholtz timescale for the Sun by assuming that it has radiated its present luminosity of $L_\odot = 4 \times 10^{33} \text{ erg s}^{-1}$ for its entire life. Then

\[ t_{KH} \approx \frac{E_{\text{rad}}}{L_\odot} \approx \frac{GM_\odot^2}{R_\odot L_\odot} \approx 10^7 \text{ years}, \]

We conclude that the Sun contracted to the main sequence on a Kelvin–Helmholtz timescale of about 10 million years.

Generally, we shall define a Kelvin–Helmholtz timescale for a star by the relation

\[ t_{KH} = \frac{\Omega L}{L} \approx \frac{GM^2}{RL}, \]

where $R$ is the radius, $M$ the mass, and $L$ the luminosity.
4.7 Kelvin–Helmholtz Timescale for Other Stars

The Kelvin–Helmholtz timescale for other stars may be related to that of the Sun by scaling. Since generally

\[ t_{KH} = \frac{\Omega}{L} \simeq \frac{GM^2}{LR}, \]

the ratio of the Kelvin–Helmholtz timescale for some star relative to that of the Sun is given by

\[ \frac{t_{KH}}{t_{\odot KH}} = \left( \frac{R_{\odot}}{R} \right) \left( \frac{L_{\odot}}{L} \right) \left( \frac{M}{M_{\odot}} \right)^2, \]

where a good estimate is \( t_{\odot KH} \approx 3 \times 10^7 \text{ yr} \).

An A0 main sequence star has

\[ R = 2.5 R_{\odot} \quad M = 3.2 M_{\odot} \quad L = 79.4 L_{\odot}. \]

From the above equation, the Kelvin–Helmholtz timescale is

\[ t_{KH} \sim 0.055 t_{\odot KH} \sim 1.55 \times 10^6 \text{ yr}. \]

More massive stars evolve more rapidly through all phases of their lives, including periods of gravitational contraction.