Supplement: The Chern Theorem and High-Spin Topological States in Nuclear Structure Physics

Mike Guidry and Yang Sun

This document provides supplemental material and proofs of some important results for the article "The Chern Theorem and High-Spin Topological States in Nuclear Structure Physics" by Mike Guidry and Yang Sun, published in the journal *xxxx* (2024).

I. INTRODUCTION

In the following all citations of sections, subsections, equation numbers, figure numbers, and table numbers are by default references to the primary document "Conjectured High-Spin Topological States in Nuclear Structure Physics" published by the present authors in the journal *xxxx* (2024). If a reference is flagged by "[this document]", it is instead a reference to objects in the present *Supplement* document.

II. THE CHERN THEOREM FOR A 2D-ROTOR HILBERT SPACE

This section contains derivations and technical details associated with Chern quantization of a nuclear quantum rotor having a 2D Hilbert space. It represents application of standard Chern-theorem techniques for topological condensed matter [2–7] to the specific problem of a nuclear quantum rotor.

A. Angular Momentum Operators

Comparing the Euler angles (β,γ) in Fig. 1(b) with the standard spherical polar angles (θ,ϕ) in Fig. 1(c), successive rotations through the Euler angles γ and then β, and successive rotations through the polar angles ϕ and then θ, are related by

$$e^{i\beta\hat{J}_y}e^{i\gamma\hat{J}_z} \longleftrightarrow e^{i\theta\hat{J}_y}e^{i\phi\hat{J}_z}.$$

Therefore, the 2D Euler-angle coordinate system in Fig. 1(b) is equivalent to the spherical polar coordinate system in Fig. 1(c) under the mapping $\theta\leftrightarrow\beta$. and $\phi\leftrightarrow\gamma$, and this mapping may be used to express formulas valid for the spherical polar coordinates (θ,ϕ) in the 2D Euler-angle space (β,γ). For example, the coordinate representation of the angular momentum operators $(\hat{J}_x,\hat{J}_y,\hat{J}_z)$ in the (β,γ) coordinates is obtained from standard formulas expressed in spherical coordinates using the mapping $(\theta,\phi)\leftrightarrow(\beta,\gamma)$ as

$$\hat{J}_x = i\hbar\left(\sin\gamma\frac{\partial}{\partial\beta}+\cot\beta\cos\gamma\frac{\partial}{\partial\gamma}\right), \tag{1a}$$

$$\hat{J}_y = i\hbar\left(-\cos\gamma\frac{\partial}{\partial\beta}+\cot\beta\cos\gamma\frac{\partial}{\partial\gamma}\right), \tag{1b}$$

$$\hat{J}_z = -i\hbar\frac{\partial}{\partial\gamma}. \tag{1c}$$

(We will often choose $\hbar=1$ units and drop the explicit factors of $\hbar$.)

Let's find expectation values of angular momentum operator components for the laboratory frame $(\hat{J}_x,\hat{J}_y,\hat{J}_z)$ and the intrinsic frame $(\hat{J}_1,\hat{J}_2,\hat{J}_3)$. Consider $\hat{J}_x$ in the lab frame; utilizing Eq. (9),

$$\langle\psi|\hat{J}_x|\psi\rangle = \langle\Phi|e^{i\gamma\hat{J}_z}e^{i\beta\hat{J}_y}\hat{J}_x e^{-i\beta\hat{J}_y}e^{-i\gamma\hat{J}_z}|\Phi\rangle. \tag{2}$$

But if the intrinsic-frame operator component $\hat{J}_1$ is defined in terms of the lab-frame operator component $\hat{J}_x$ by

$$\hat{J}_1 = \hat{D}^{-1}\hat{J}_x\hat{D} = e^{i\gamma\hat{J}_z}e^{i\beta\hat{J}_y}\hat{J}_x e^{-i\beta\hat{J}_y}e^{-i\gamma\hat{J}_z}, \tag{3}$$

then $\langle\psi|\hat{J}_x|\psi\rangle = \langle\Phi|\hat{J}_1|\Phi\rangle$. and from Eqs. (2)-(3) [this document],

$$\langle\psi|\hat{J}_x|\psi\rangle = \langle\Phi|e^{i\gamma\hat{J}_z}e^{i\beta\hat{J}_y}\hat{J}_x e^{-i\beta\hat{J}_y}e^{-i\gamma\hat{J}_z}|\Phi\rangle = \langle\Phi|\hat{J}_1|\Phi\rangle, \tag{4}$$

where $\hat{J}_x$ and $|\psi\rangle$ are lab-frame and $\hat{J}_1$ and $|\Phi\rangle$ are intrinsic-frame quantities. Now consider $\hat{J}_y$ in the lab frame,

$$\begin{aligned}\langle\psi|\hat{J}_y|\psi\rangle &= \langle\Phi|e^{i\gamma\hat{J}_z}e^{i\beta\hat{J}_y}\hat{J}_y e^{-i\beta\hat{J}_y}e^{-i\gamma\hat{J}_z}|\Phi\rangle \\ &= \langle\Phi|e^{i\gamma\hat{J}_z}e^{i\beta\hat{J}_y}e^{-i\beta\hat{J}_y}\hat{J}_y e^{-i\gamma\hat{J}_z}|\Phi\rangle \\ &= \langle\Phi|e^{i\gamma\hat{J}_z}\hat{J}_y e^{-i\gamma\hat{J}_z}|\Phi\rangle,\end{aligned} \tag{5}$$

where we have used that $[\hat{A},e^{\hat{B}}] = 0$ only if $[\hat{A},\hat{B}] = 0$. But defining the intrinsic frame operator component $\hat{J}_2 \equiv \hat{D}^{-1}\hat{J}_y\hat{D}$ in terms of the lab frame component $\hat{J}_y$,

$$\hat{J}_2 \equiv e^{i\gamma\hat{J}_z}e^{i\beta\hat{J}_y}\hat{J}_y e^{-i\beta\hat{J}_y}e^{-i\gamma\hat{J}_z} = e^{i\gamma\hat{J}_z}\hat{J}_y e^{-i\gamma\hat{J}_z}, \tag{6}$$

and using $\langle\psi|\hat{J}_y|\psi\rangle = \langle\Phi|\hat{J}_2|\Phi\rangle$ and Eq. (5) [this document],

$$\langle\psi|\hat{J}_y|\psi\rangle = \langle\Phi|e^{i\gamma\hat{J}_z}\hat{J}_y e^{-i\gamma\hat{J}_z}|\Phi\rangle = \langle\Phi|\hat{J}_2|\Phi\rangle, \tag{7}$$

where $\hat{J}_y$ and $|\psi\rangle$ are defined in the lab frame, and $\hat{J}_2$ and $|\Phi\rangle$ are defined in the intrinsic frame. Finally, consider the expectation value of $\hat{J}_z$ evaluated in the laboratory frame,

$$\langle\psi|\hat{J}_z|\psi\rangle = \langle\Phi|e^{i\gamma\hat{J}_z}e^{i\beta\hat{J}_y}\hat{J}_z e^{-i\beta\hat{J}_y}e^{-i\gamma\hat{J}_z}|\Phi\rangle. \tag{8}$$

If the intrinsic-frame operator component $\hat{J}_3$ is defined by

$$\hat{J}_3 \equiv \hat{D}^{-1}\hat{J}_z\hat{D} = e^{i\gamma\hat{J}_z}e^{i\beta\hat{J}_y}\hat{J}_z e^{-i\beta\hat{J}_y}e^{-i\gamma\hat{J}_z}, \tag{9}$$

then $\langle\psi|\hat{J}_z|\psi\rangle = \langle\Phi|\hat{J}_3|\Phi\rangle$ and from Eqs. (8)-(9) [this document],

$$\langle\psi|\hat{J}_z|\psi\rangle = \langle\Phi|e^{i\gamma\hat{J}_z}e^{i\beta\hat{J}_y}\hat{J}_z e^{-i\beta\hat{J}_y}e^{-i\gamma\hat{J}_z}|\Phi\rangle = \langle\Phi|\hat{J}_3|\Phi\rangle, \tag{10}$$

with $\hat{J}_z$ and $|\psi\rangle$ laboratory-frame and $\hat{J}_3$ and $|\Phi\rangle$ intrinsic-frame quantities.

B. Important Commutation Relations

In the derivations in this paper it will often be necessary to commute operators in expressions. In doing so we will use extensively that an operator $\hat{A}$ commutes with a function $f(\hat{A})$ of that operator

$$[\hat{A}, f(\hat{A})] = 0. \tag{11}$$

For example, Eq. (1c) [this document] in $\hbar = 1$ units implies that $e^{-i\gamma\hat{J}_z}$ is a function of the derivative operator $\frac{\partial}{\partial\gamma}$,

$$e^{-i\gamma\hat{J}_z} = e^{-\gamma\frac{\partial}{\partial\gamma}} = f(\tfrac{\partial}{\partial\gamma}), \tag{12}$$

and therefore $e^{-i\gamma\hat{J}_z}$ commutes with $\frac{\partial}{\partial\gamma}$,

$$[e^{-i\gamma\hat{J}_z}, \tfrac{\partial}{\partial\gamma}] = 0. \tag{13}$$

We will also use that an operator $\hat{A}$ commutes with the exponentiation of an operator $\hat{B}$ if and only if $\hat{A}$ and $\hat{B}$ commute:

$$[\hat{A}, e^{\hat{B}}] = 0 \qquad (\text{iff } [\hat{A},\hat{B}] = 0). \tag{14}$$

For example Eq. (13) [this document] implies that J_z commutes with $e^{i\gamma J_z}$. We will use often that the derivative of an exponential of an operator can be written with either the residual exponential or the derivative of the exponent to the right. For example,

$$\frac{\partial}{\partial\beta}\left(e^{-i\beta\hat{J}_y}\right) = i\,\frac{\partial(\beta\hat{J}_y)}{\partial\beta}\,e^{-i\beta\hat{J}_y} = i\,e^{-i\beta\hat{J}_y}\,\frac{\partial(\beta\hat{J}_y)}{\partial\beta}, \tag{15}$$

which follows from taking the derivative of the expression $[\hat{J}_y, e^{-i\beta\hat{J}_y}] = 0$, which is true because of Eq. (11) [this document]. Finally we will often use that since the intrinsic state $|\Phi\rangle$ is assumed to be independent of the Euler angles (β, γ), any derivative operator with respect to these variables gives zero when applied to $|\Phi\rangle$,

$$\frac{\partial}{\partial\beta}|\Phi\rangle = \frac{\partial}{\partial\gamma}|\Phi\rangle = \frac{\partial^2}{\partial\beta^2}|\Phi\rangle = \frac{\partial^2}{\partial\gamma^2}|\Phi\rangle = \frac{\partial^2}{\partial\beta\partial\gamma}|\Phi\rangle = \frac{\partial^2}{\partial\gamma\partial\beta}|\Phi\rangle = 0. \tag{16}$$

This will allow elimination of terms involving derivatives if the derivative operator can be commuted to the right to act on the intrinsic state.

C. Some Derivatives

Let's calculate some derivatives that will be needed, assuming that the Liebniz relation for derivatives of products

$$\frac{d}{dx}(AB) = \frac{d}{dx}(A)B + A\frac{d}{dx}(B) \tag{17}$$

holds also for possibly non-commuting operators. Then,

$$\begin{aligned}\frac{\partial}{\partial\beta}\left(e^{-i\beta\hat{J}_y}\right) &= -ie^{-i\beta\hat{J}_y}\frac{\partial}{\partial\beta}(\beta\hat{J}_y) = -ie^{-i\beta\hat{J}_y}\left(\hat{J}_y + \beta\frac{\partial\hat{J}_y}{\partial\beta}\right)\\ &= -ie^{-i\beta\hat{J}_y}\left[\hat{J}_y - i\beta\left(\cos\gamma\frac{\partial^2}{\partial\beta^2} + \frac{\sin\gamma}{\sin^2\beta}\frac{\partial}{\partial\gamma}\right)\right],\end{aligned} \tag{18}$$

where we have used that

$$\begin{aligned}\frac{\partial\hat{J}_y}{\partial\beta} &= i\frac{\partial}{\partial\beta}\left(-\cos\gamma\frac{\partial}{\partial\beta} + \cot\beta\cos\gamma\frac{\partial}{\partial\gamma}\right)\\ &= -i\left(\cos\gamma\frac{\partial^2}{\partial\beta^2} + \frac{\sin\gamma}{\sin^2\beta}\frac{\partial}{\partial\gamma}\right),\end{aligned} \tag{19}$$

from Eq. (1b) [this document]. In a similar manner,

$$\begin{aligned}\frac{\partial}{\partial\beta}\left(e^{-i\gamma\hat{J}_z}\right) &= ie^{-i\gamma\hat{J}_z}\frac{\partial}{\partial\beta}(\gamma\hat{J}_z) = i\gamma e^{-i\gamma\hat{J}_z}\frac{\partial J_z}{\partial\beta}\\ &= i\gamma e^{-i\gamma\hat{J}_z}\frac{\partial}{\partial\beta}\left(-i\frac{\partial}{\partial\gamma}\right) = \gamma e^{-i\gamma\hat{J}_z}\frac{\partial^2}{\partial\beta\partial\gamma},\end{aligned} \tag{20}$$

where we have used Eq. (1c) [this document] in line 2. Likewise,

$$\begin{aligned}\frac{\partial}{\partial\gamma}\left(e^{-i\beta\hat{J}_y}\right) &= -ie^{-i\beta\hat{J}_j}\frac{\partial}{\partial\gamma}(\beta\hat{J}_y) = -ie^{-i\beta\hat{J}_j}\beta\frac{\partial\hat{J}_y}{\partial\gamma}\\ &= e^{-i\beta\hat{J}_j}\beta\left(\sin\gamma\frac{\partial}{\partial\beta} + \cot\beta\cos\gamma\frac{\partial}{\partial\gamma} + \cot\beta\sin\gamma\frac{\partial^2}{\partial\gamma^2}\right),\end{aligned} \tag{21}$$

where we have used Eq. (1b) [this document] to calculate the derivative

$$\frac{\partial\hat{J}_y}{\partial\gamma} = i\left[\sin\gamma\frac{\partial}{\partial\beta} + \cot\beta\left(\cos\gamma\frac{\partial}{\partial\gamma} + \sin\gamma\frac{\partial^2}{\partial\gamma^2}\right)\right]. \tag{22}$$

In a similar manner,

$$\frac{\partial}{\partial\gamma}\left(e^{-i\gamma\hat{J}_z}\right) = -ie^{-i\gamma\hat{J}_z}\frac{\partial}{\partial\gamma}(\gamma\hat{J}_y) = -ie^{-i\gamma\hat{J}_z}\left(\hat{J}_z - i\gamma\frac{\partial^2}{\partial\gamma^2}\right), \tag{23}$$

where we have used

$$\frac{\partial}{\partial\gamma}(\gamma\hat{J}_z) = \hat{J}_z + \gamma\frac{\partial\hat{J}_z}{\partial\gamma} = \hat{J}_z - i\gamma\frac{\partial^2}{\partial\gamma^2}, \tag{24}$$

which was deduced from Eq. (1c) [this document] upon setting $\hbar = 1$.

D. Berry Connections

The Berry connections for the Hilbert space of Eq. (9) are

$$A_\beta \equiv \langle \psi | i\partial_\beta | \psi \rangle \qquad A_\gamma \equiv \langle \psi | i\partial_\gamma | \psi \rangle, \tag{25}$$

where $\partial_\alpha \equiv \partial/\partial\alpha$. The wavefunctions $|\beta\gamma\rangle \equiv |\psi(\beta,\gamma)\rangle$ are

$$|\beta\gamma\rangle = e^{-i\beta\hat{J}_y} e^{-i\gamma\hat{J}_z} |\Phi\rangle \qquad \langle\beta\gamma| = \langle\Phi| e^{i\gamma\hat{J}_z} e^{i\beta\hat{J}_y}, \tag{26}$$

Using these wavefunctions, we then have

$$\begin{aligned}
\frac{\partial}{\partial\gamma}|\beta\gamma\rangle &= \frac{\partial}{\partial\gamma}\left(e^{-i\beta\hat{J}_y}e^{-i\gamma\hat{J}_z}\right)|\Phi\rangle = \frac{\partial}{\partial\gamma}\left(e^{-i\beta\hat{J}_y}\right)e^{-i\gamma\hat{J}_z}|\Phi\rangle + e^{-i\beta\hat{J}_y}\frac{\partial}{\partial\gamma}\left(e^{-i\gamma\hat{J}_z}\right)|\Phi\rangle \\
&= e^{-i\beta\hat{J}_y}\beta\left(\sin\gamma\frac{\partial}{\partial\beta} + \cot\beta\cos\gamma\frac{\partial}{\partial\gamma} + \cot\beta\sin\gamma\frac{\partial^2}{\partial\gamma^2}\right)e^{-i\gamma\hat{J}_z}|\Phi\rangle - ie^{-i\beta\hat{J}_y}e^{-i\gamma\hat{J}_z}\left(\hat{J}_z - i\gamma\frac{\partial^2}{\partial\gamma^2}\right)|\Phi\rangle \\
&= -ie^{-i\beta\hat{J}_y}\hat{J}_z e^{-i\gamma\hat{J}_z}|\Phi\rangle,
\end{aligned} \tag{27}$$

where in lines 2-3 manipulations like the following have been used to set all terms containing derivative operators to zero:

$$\begin{aligned}
\sin\gamma\frac{\partial}{\partial\beta}\left(e^{-i\gamma\hat{J}_z}\right)|\Phi\rangle &= \sin\gamma\, e^{-i\gamma\hat{J}_z}\frac{\partial}{\partial\beta}(\gamma\hat{J}_z)|\Phi\rangle \\
&= \sin\gamma\, e^{-i\gamma\hat{J}_z}\gamma\frac{\partial}{\partial\beta}\left(-i\frac{\partial}{\partial\gamma}\right)|\Phi\rangle \\
&= -i\gamma\sin\gamma\, e^{-i\gamma\hat{J}_z}\frac{\partial^2}{\partial\beta\partial\gamma}|\Phi\rangle = 0,
\end{aligned}$$

where Eqs. (1c), (15), and (16) [this document] have been used. By similar reasoning,

$$\cot\beta\cos\gamma\frac{\partial}{\partial\gamma}e^{-i\gamma\hat{J}_z}|\Phi\rangle = 0, \tag{28}$$

where we have used that from Eqs. (16) and (15) [this document],

$$\frac{\partial}{\partial\gamma}e^{-i\gamma\hat{J}_z}|\Phi\rangle = e^{-i\gamma\hat{J}_z}\frac{\partial}{\partial\gamma}|\Phi\rangle = 0. \tag{29}$$

Finally we have

$$\cot\beta\sin\gamma\frac{\partial^2}{\partial\gamma^2}e^{-i\gamma\hat{J}_z}|\Phi\rangle = \cot\beta\sin\gamma\frac{\partial}{\partial\gamma}\left(e^{-i\gamma\hat{J}_z}\frac{\partial}{\partial\gamma}\right)|\Phi\rangle = 0, \tag{30}$$

by virtue of Eq. (29) [this document], and

$$i\gamma\frac{\partial^2}{\partial\gamma^2}|\Phi\rangle = 0, \tag{31}$$

because of Eq. (16) [this document]. Therefore, all terms in Eq. (27) [this document] involving derivative operators vanish and we are left with

$$\frac{\partial}{\partial\gamma}|\beta\gamma\rangle = -ie^{-i\beta\hat{J}_y}\hat{J}_z e^{-i\gamma\hat{J}_z}|\Phi\rangle. \tag{32}$$

In a similar manner we find

$$\begin{aligned}
\frac{\partial}{\partial\beta}|\beta\gamma\rangle &= \frac{\partial}{\partial\beta}\left(e^{-i\beta\hat{J}_y}e^{-i\gamma\hat{J}_z}\right)|\Phi\rangle \\
&= \left[\frac{\partial}{\partial\beta}\left(e^{-i\beta\hat{J}_y}\right)e^{-i\gamma\hat{J}_z} + e^{-i\beta\hat{J}_y}\frac{\partial}{\partial\beta}\left(e^{-i\gamma\hat{J}_z}\right)\right]|\Phi\rangle \\
&= -ie^{-i\beta\hat{J}_y}\left[\hat{J}_y - i\beta\left(\cos\gamma\frac{\partial^2}{\partial\beta^2} + \frac{\sin\gamma}{\sin^2\beta}\frac{\partial}{\partial\gamma}\right)\right] \\
&= -i\hat{J}_y e^{-i\beta\hat{J}_y}e^{-i\gamma\hat{J}_z}|\Phi\rangle.
\end{aligned} \tag{33}$$

The results in Eqs. (27) and (33) [this document] may then be used to calculate the Berry connections A_β and A_γ from

$$A_\beta = \langle \beta\gamma | i\tfrac{\partial}{\partial\beta} | \beta\gamma\rangle \qquad A_\gamma = \langle \beta\gamma | i\tfrac{\partial}{\partial\gamma} | \beta\gamma\rangle .$$

From Eq. (33) [this document]

$$\frac{\partial}{\partial\beta}|\beta\gamma\rangle = -i\hat{J}_y e^{-i\beta\hat{J}_y} e^{-i\gamma\hat{J}_z}|\Phi\rangle, \tag{34}$$

$$\frac{\partial}{\partial\gamma}|\beta\gamma\rangle = -ie^{-i\beta\hat{J}_y}\hat{J}_z e^{-i\gamma\hat{J}_z}|\Phi\rangle, \tag{35}$$

where $|\Phi\rangle$ is assumed not to depend on the Euler angles (β, γ). The connection A_β is given by

$$\begin{aligned} A_\beta = \langle \beta\gamma | i\tfrac{\partial}{\partial\beta} | \beta\gamma\rangle &= i(-i)\langle\Phi|e^{i\gamma\hat{J}_z}e^{i\beta\hat{J}_y}\hat{J}_y e^{-i\beta\hat{J}_y}e^{-i\gamma\hat{J}_z}|\Phi\rangle \\ &= \langle\Phi|e^{i\gamma\hat{J}_z}e^{i\beta\hat{J}_y}e^{-i\beta\hat{J}_y}\hat{J}_y e^{-i\gamma\hat{J}_z}|\Phi\rangle \\ &= \langle\Phi|e^{i\gamma\hat{J}_z}\hat{J}_y e^{-i\gamma\hat{J}_z}|\Phi\rangle = \langle\Phi|\hat{J}_2|\Phi\rangle, \end{aligned} \tag{36}$$

where we've used Eq. (34) [this document], that $[\hat{A}, e^{\hat{B}}] = 0$ only if $[\hat{A}, \hat{B}] = 0$ from Eq. (14) [this document], and Eq. (6) [this document]. Likewise, the connection A_γ is

$$\begin{aligned} A_\gamma = \langle \beta\gamma | i\tfrac{\partial}{\partial\gamma} | \beta\gamma\rangle &= i(-i)\langle\Phi|e^{i\gamma\hat{J}_z}e^{i\beta\hat{J}_y}e^{-i\beta\hat{J}_y}\hat{J}_z e^{-i\gamma\hat{J}_z}|\Phi\rangle \\ &= \langle\Phi|e^{i\gamma\hat{J}_z}\hat{J}_z e^{-i\gamma\hat{J}_z}|\Phi\rangle = \langle\Phi|\hat{J}_z|\Phi\rangle, \end{aligned} \tag{37}$$

where Eq. (35) [this document] was used.

E. Berry Curvature

From Eq. (15), we need $\partial A_\beta/\partial\gamma$ to evaluate the integral of Berry curvature over Hilbert space. From Eq. (36) [this document],

$$\begin{aligned} \frac{\partial A_\beta}{\partial\gamma} &= \frac{\partial}{\partial\gamma}\langle\Phi|e^{i\gamma\hat{J}_z}\hat{J}_y e^{-i\gamma\hat{J}_z}|\Phi\rangle \\ &= \left\langle \Phi \left| \frac{\partial}{\partial\gamma}\left(e^{i\gamma\hat{J}_z}\right)\hat{J}_y e^{-i\gamma\hat{J}_z} \right| \Phi \right\rangle \qquad \text{(Term 1)} \\ &+ \left\langle \Phi \left| e^{i\gamma\hat{J}_z}\frac{\partial\hat{J}_y}{\partial\gamma} e^{-i\gamma\hat{J}_z} \right| \Phi \right\rangle \qquad \text{(Term 2)} \\ &+ \left\langle \Phi \left| e^{i\gamma\hat{J}_z}\hat{J}_y \frac{\partial}{\partial\gamma}\left(e^{-i\gamma\hat{J}_z}\right) \right| \Phi \right\rangle \qquad \text{(Term 3)}, \end{aligned} \tag{38}$$

where $|\Phi\rangle$ is assumed to be independent of (β, γ). Let's now evaluate each of the three terms in Eq. (38) [this document]. For the first term,

$$\begin{aligned} \text{Term 1} &= \left\langle \Phi \left| \frac{\partial}{\partial\gamma}\left(e^{i\gamma\hat{J}_z}\right)\hat{J}_y e^{-i\gamma\hat{J}_z} \right| \Phi \right\rangle \\ &= \left\langle \Phi \left| ie^{i\gamma\hat{J}_z}\left(\hat{J}_z - \gamma\frac{\partial^2}{\partial\gamma^2}\right)\hat{J}_y e^{-i\gamma\hat{J}_z} \right| \Phi \right\rangle \\ &= \left\langle \Phi \left| ie^{i\gamma\hat{J}_z}\hat{J}_z\hat{J}_y e^{-i\gamma\hat{J}_z} \right| \Phi \right\rangle + \left\langle \Phi \left| e^{i\gamma\hat{J}_z}\gamma\frac{\partial^2}{\partial\gamma^2}\hat{J}_y e^{-i\gamma\hat{J}_z} \right| \Phi \right\rangle \\ &= \left\langle \Phi \left| ie^{i\gamma\hat{J}_z}\hat{J}_z\hat{J}_y e^{-i\gamma\hat{J}_z} \right| \Phi \right\rangle + \left\langle \Phi \left| e^{i\gamma\hat{J}_z}\gamma\frac{\partial}{\partial\gamma}\left[\frac{\partial}{\partial\gamma}\left(\hat{J}_y e^{-i\gamma\hat{J}_z}\right)\right] \right| \Phi \right\rangle \\ &= \left\langle \Phi \left| ie^{i\gamma\hat{J}_z}\hat{J}_z\hat{J}_y e^{-i\gamma\hat{J}_z} \right| \Phi \right\rangle, \end{aligned} \tag{39}$$

where in line 2 Eq. (23) [this document] was used and the second term of line 4 was eliminated by

$$\begin{aligned}
\left\langle \Phi \left| e^{i\gamma\hat{J}_z}\gamma\frac{\partial}{\partial\gamma}\left[\frac{\partial}{\partial\gamma}\left(\hat{J}_y e^{-i\gamma\hat{J}_z}\right)\right]\right| \Phi \right\rangle &= \left\langle \Phi \left| e^{i\gamma\hat{J}_z}\gamma\frac{\partial}{\partial\gamma}\left[\left(\frac{\partial\hat{J}_y}{\partial\gamma}\right)e^{-i\gamma\hat{J}_z} + \hat{J}_y\frac{\partial}{\partial\gamma}\left(e^{-i\gamma\hat{J}_z}\right)\right]\right| \Phi \right\rangle \\
&= \left\langle \Phi \left| e^{i\gamma\hat{J}_z}\gamma\frac{\partial}{\partial\gamma}\left[\left(\frac{\partial\hat{J}_y}{\partial\gamma}\right)e^{-i\gamma\hat{J}_z} + \hat{J}_y\left(e^{-i\gamma\hat{J}_z}\right)\frac{\partial}{\partial\gamma}\right]\right| \Phi \right\rangle \\
&= \left\langle \Phi \left| e^{i\gamma\hat{J}_z}\gamma\frac{\partial}{\partial\gamma}\left[\left(\frac{\partial\hat{J}_y}{\partial\gamma}\right)e^{-i\gamma\hat{J}_z}\right]\right| \Phi \right\rangle \\
&= \left\langle \Phi \left| e^{i\gamma\hat{J}_z} i\gamma\frac{\partial}{\partial\gamma}\left[\left(\sin\gamma\frac{\partial}{\partial\beta} + \cot\beta\cos\gamma\frac{\partial}{\partial\gamma} + \cot\beta\sin\gamma\frac{\partial^2}{\partial\gamma^2}\right)e^{-i\gamma\hat{J}_z}\right]\right| \Phi \right\rangle = 0, \qquad (40)
\end{aligned}$$

where in line 2 we have used Eq. (13) [this document], in line 3 we have used Eq. (16) [this document], in line 4 we have used Eq. (16) [this document], and in line 4 the partial derivative factors can all be brought to the right to act on $|\Phi\rangle$ and give zero by commuting with the exponential factor $\exp(-i\gamma\hat{J}_z)$:

- The exponential $\exp(-i\gamma\hat{J}_z)$ does not depend on β so it commutes with $\partial/\partial\beta$.
- From Eq. (13) [this document], $\exp(-i\gamma\hat{J}_z)$ commutes with $\partial/\partial\gamma$.
- In the last term of line 5,

$$\frac{\partial^2}{\partial\gamma^2}e^{-i\gamma\hat{J}_z} = \frac{\partial}{\partial\gamma}\left[\frac{\partial}{\partial\gamma}e^{-i\gamma\hat{J}_z}\right] = \frac{\partial}{\partial\gamma}\left[e^{-i\gamma\hat{J}_z}\frac{\partial}{\partial\gamma}\right]$$

  where Eq. (13) [this document] was used.

From Eq. (16) [this document], the partial derivative factors then all give zero when operating on the intrinsic state $|\Phi\rangle$ and the entire contribution of Eq. (40) [this document] to Term 1 vanishes. By similar means, Term 2 vanishes identically,

$$\begin{aligned}
\text{Term 2} &= \left\langle \Phi \left| e^{i\gamma\hat{J}_z}\left(\frac{\partial\hat{J}_y}{\partial\gamma}\right)e^{-i\gamma\hat{J}_z}\right| \Phi \right\rangle \\
&= \left\langle \Phi \left| e^{i\gamma\hat{J}_z} i\left(\sin\gamma\frac{\partial}{\partial\beta} + \cot\beta\cos\gamma\frac{\partial}{\partial\gamma} + \cot\beta\sin\gamma\frac{\partial^2}{\partial\gamma^2}\right)e^{-i\gamma\hat{J}_z}\right| \Phi \right\rangle \\
&= \left\langle \Phi \left| e^{i\gamma\hat{J}_z} i\left(\sin\gamma e^{-i\gamma\hat{J}_z}\frac{\partial}{\partial\beta} + \cot\beta\cos\gamma e^{-i\gamma\hat{J}_z}\frac{\partial}{\partial\gamma} + \cot\beta\sin\gamma\frac{\partial}{\partial\gamma}\left[e^{-i\gamma\hat{J}_z}\frac{\partial}{\partial\gamma}\right]\right)\right| \Phi \right\rangle = 0, \qquad (41)
\end{aligned}$$

where Eq. (22) [this document] was used to expand $\frac{\partial\hat{J}_y}{\partial\gamma}$ in line 2, and in line 3 the partial derivative factors can all be brought to the right to act on $|\Phi\rangle$ by commuting with the exponential factor $\exp(-i\gamma\hat{J}_z)$ on their right in the same manner as for Term 1:

- The exponential $\exp(-i\gamma\hat{J}_z)$ does not depend on β so it commutes with $\partial/\partial\beta$.
- From Eq. (13) [this document], $\exp(-i\gamma\hat{J}_z)$ commutes with $\partial/\partial\gamma$.
- In the last term of line 2,

$$\frac{\partial^2}{\partial\gamma^2}e^{-i\gamma\hat{J}_z} = \frac{\partial}{\partial\gamma}\left[\frac{\partial}{\partial\gamma}e^{-i\gamma\hat{J}_z}\right] = \frac{\partial}{\partial\gamma}\left[e^{-i\gamma\hat{J}_z}\frac{\partial}{\partial\gamma}\right],$$

  where Eq. (13) [this document] was used.

From Eq. (16) [this document], the partial derivative factors then all give zero when operating on the intrinsic state $|\Phi\rangle$ and Term 2 vanishes. Proceeding in a similar way, Term 3 gives

$$\begin{aligned}
\text{Term 3} &= \left\langle \Phi \left| e^{i\gamma\hat{J}_z}\hat{J}_y\frac{\partial}{\partial\gamma}\left(e^{-i\gamma\hat{J}_z}\right)\right| \Phi \right\rangle \\
&= \left\langle \Phi \left| e^{i\gamma\hat{J}_z}\hat{J}_y(-i)\left(\hat{J}_z - i\gamma\frac{\partial^2}{\partial\gamma^2}\right)e^{-i\gamma\hat{J}_z}\right| \Phi \right\rangle \\
&= -i\left\langle \Phi \left| e^{i\gamma\hat{J}_z}\hat{J}_y\hat{J}_z e^{-i\gamma\hat{J}_z}\right| \Phi \right\rangle - \left\langle \Phi \left| e^{i\gamma\hat{J}_z}\hat{J}_y\gamma\frac{\partial^2}{\partial\gamma^2}e^{-i\gamma\hat{J}_z}\right| \Phi \right\rangle \\
&= -i\left\langle \Phi \left| e^{i\gamma\hat{J}_z}\hat{J}_y\hat{J}_z e^{-i\gamma\hat{J}_z}\right| \Phi \right\rangle - \left\langle \Phi \left| e^{i\gamma\hat{J}_z}\hat{J}_y\gamma\frac{\partial}{\partial\gamma}\left(e^{-i\gamma\hat{J}_z}\frac{\partial}{\partial\gamma}\right)\right| \Phi \right\rangle \\
&= -i\langle\Phi|e^{i\gamma\hat{J}_z}\hat{J}_y\hat{J}_z e^{-i\gamma\hat{J}_z}|\Phi\rangle, \qquad (42)
\end{aligned}$$

where Eq. (13) [this document] was used in the second term of line 3 to bring $\partial/\partial\gamma$ to the right and the second term in line 4 vanishes because of Eq. (16) [this document]. Inserting the results (39), (41), and (42) [this document] into Eq. (38) [this document] leads to

$$\begin{aligned}
\frac{\partial A_\beta}{\partial\gamma} &= \frac{\partial}{\partial\gamma}\langle\Phi|e^{i\gamma\hat{J}_z}\hat{J}_y e^{-i\gamma\hat{J}_z}|\Phi\rangle \\
&= \langle\Phi|i\hat{J}_z e^{i\gamma\hat{J}_z}\hat{J}_y e^{-i\gamma\hat{J}_z}|\Phi\rangle - i\langle\Phi|e^{i\gamma\hat{J}_z}\hat{J}_y\hat{J}_z e^{-i\gamma\hat{J}_z}|\Phi\rangle \\
&= i\langle\Phi|e^{i\gamma\hat{J}_z}\hat{J}_z\hat{J}_y e^{-i\gamma\hat{J}_z}|\Phi\rangle - i\langle\Phi|e^{i\gamma\hat{J}_z}\hat{J}_y\hat{J}_z e^{-i\gamma\hat{J}_z}|\Phi\rangle \\
&= i\langle\Phi|e^{i\gamma\hat{J}_z}(\hat{J}_z\hat{J}_y - \hat{J}_y\hat{J}_z)e^{-i\gamma\hat{J}_z}|\Phi\rangle \\
&= i\langle\Phi|e^{i\gamma\hat{J}_z}\,[\hat{J}_z,\hat{J}_y]\,e^{-i\gamma\hat{J}_z}|\Phi\rangle \\
&= \langle\Phi|e^{i\gamma\hat{J}_z}\hat{J}_x e^{-i\gamma\hat{J}_z}|\Phi\rangle,
\end{aligned} \tag{43}$$

where in the second line we have used the usual formula for the derivative of a product and in the last line we have used that the $\hat{J}_i$ are generators of angular momentum that obey an SU(2) Lie algebra $[\hat{J}_i,\hat{J}_j] = i\varepsilon_{ijk}\hat{J}_k$, where ε_{ijk} is the completely antisymmetric rank-2 tensor. Thus, integration of the Berry curvature over the Hilbert space has recovered the SU(2) Lie algebra of the rotor angular momentum operators.

Then the Berry curvature integrated over the Hilbert-space manifold $\Gamma^{\gamma\beta}$ is

$$\begin{aligned}
\Gamma^{\gamma\beta} \equiv \int_S \Omega\, dA &= \int_{\gamma_i}^{\gamma_f} d\gamma \int_0^\pi d\beta \sin\beta \left(\frac{\partial A_\gamma}{\partial\beta} - \frac{\partial A_\beta}{\partial\gamma}\right) \\
&= -\int_{\gamma_i}^{\gamma_f} d\gamma \int_0^\pi d\beta \sin\beta \left(\frac{\partial A_\beta}{\partial\gamma}\right) \\
&= -\int_{\gamma_i}^{\gamma_f} d\gamma \frac{\partial}{\partial\gamma}\int_0^\pi d\beta \sin\beta A_\beta \\
&= -\int_{\gamma_i}^{\gamma_f} d\gamma \frac{\partial\phi^\beta}{\partial\gamma} \\
&= -\int_{\gamma_i}^{\gamma_f} d\phi^\beta \\
&= \phi^\beta(\gamma_i) - \phi^\beta(\gamma_f),
\end{aligned} \tag{44}$$

where in line 3 the order of integration and differentiation has been switched, and the Berry phase $\phi^\beta(\gamma)$ evaluated on a closed path in β at fixed γ has been defined by

$$\begin{aligned}
\phi^\beta(\gamma) &\equiv \int_0^\pi d\beta \sin\beta A_\beta \\
&= \langle\Phi|e^{i\gamma\hat{J}_z}\hat{J}_y e^{-i\gamma\hat{J}_z}|\Phi\rangle \int_0^\pi d\beta \sin\beta \\
&= 2\langle\Phi|e^{i\gamma\hat{J}_z}\hat{J}_y e^{-i\gamma\hat{J}_z}|\Phi\rangle = 2\langle\Phi|\hat{J}_2|\Phi\rangle,
\end{aligned} \tag{45}$$

where Eq. (36) [this document] has been used and in the second step the intrinsic state $|\Phi\rangle$ has been assumed to be independent of β. This result gives Eq. (18) of the main manuscript.

From Eq. (16), the volume of the 2D Hilbert space is $V_{\rm H} = 4\pi$. Thus, dividing both sides of Eq. (20) by $V_{\rm H}$, the Chern theorem takes the form

$$\begin{aligned}
\overline{\Gamma}^{\gamma\beta} \equiv \frac{\Gamma^{\gamma\beta}}{V_{\rm H}} &= \frac{2\langle\Phi|e^{i\gamma_i\hat{J}_z}\hat{J}_y e^{-i\hat{J}_z}|\Phi\rangle - 2\langle\Phi|e^{i\gamma_f\hat{J}_z}\hat{J}_y e^{-i\gamma_f\hat{J}_z}|\Phi\rangle}{4\pi} \\
&= \frac{1}{2\pi}\left[\big\langle\langle\Phi|e^{i\gamma_i\hat{J}_z}\hat{J}_y e^{-i\gamma_i\hat{J}_z}|\Phi\rangle\big\rangle - \big\langle\langle\Phi|e^{i\gamma_f\hat{J}_z}\hat{J}_y e^{-i\gamma_f\hat{J}_z}|\Phi\rangle\big\rangle\right] \\
&= \frac{1}{2}C_1 = \pm\frac{1}{2}\times\{0,1,2,3,\ldots\}\hbar,
\end{aligned} \tag{46}$$

where C_1 is a first Chern number, the double-bracket notation

$$\big\langle\langle A|B|C\rangle\big\rangle \equiv \frac{\langle A|B|C\rangle}{V_{\rm H}} \tag{47}$$

indicates an average of a matrix element $\langle A|B|C\rangle$ over the Hilbert-space volume $V_{\rm H}$, and we have restored a suppressed factor of $\hbar$. This gives Eq. (22) of the main manuscript.