Chapter 5: Solutions to Odd-Numbered Exercises

5.1 Expand the rotation operator in Eq. (5.2) to prove Eq. (5.5). Then let $X_\mu = J_3$ in Eq. (5.5) to give

$$\begin{aligned}
[J_3, T(LM)] &= \sum_{M'} \langle LM'|J_3|LM\rangle T(LM') \\
&= \sum_{M'} M\delta_{MM'} T(LM') \\
&= MT(LM),
\end{aligned}$$

which is Eq. (5.7) with some notation change. Likewise, setting $X_\mu = J_\pm$ in Eq. (5.5),

$$\begin{aligned}
[J_\pm, T(LM)] &= \sum_{M'} \langle LM'|J_\pm|LM\rangle T(LM') \\
&= \sum_{M'} \sqrt{L(L+1) - M(M\pm 1)}\, \delta_{M',M\pm 1} T(LM') \\
&= \sqrt{L(L+1) - M(M\pm 1)}\, T(LM),
\end{aligned}$$

which is Eq. (5.8) in different notation.

5.3 Proceed by analogy with Eqns. (5.15)–(5.22).

5.5 See Eqns. (5.54)–(5.56) and use the formula for $W(a,a,b,b;1,c)$ given in Table 5.1.

5.7 We have that

$$Q \equiv \sqrt{\frac{16\pi}{5}}\, \langle J, M=J|Q_{20}|J, M=J\rangle = \sqrt{\frac{16\pi}{5}} \begin{pmatrix} J & 2 & J \\ -J & 0 & J \end{pmatrix} \langle J\|Q^{(2)}\|J\rangle.$$

Therefore,

$$\langle J\|Q^{(2)}\|J\rangle = \begin{pmatrix} J & 2 & J \\ -J & 0 & J \end{pmatrix}^{-1} \sqrt{\frac{5}{16\pi}}\, Q.$$

Evaluating $\langle Q_{20}\rangle$ in an arbitrary state $|jm\rangle$,

$$\begin{aligned}
\sqrt{\frac{16\pi}{5}}\, \langle jm|Q_{20}|jm\rangle &= (-1)^{j-m} \sqrt{\frac{16\pi}{5}} \begin{pmatrix} j & 2 & j \\ -m & 0 & m \end{pmatrix} \langle j\|Q^{(2)}\|j\rangle \\
&= (-1)^{j-m} \sqrt{\frac{16\pi}{5}} \begin{pmatrix} j & 2 & j \\ -m & 0 & m \end{pmatrix} \sqrt{\frac{5}{16\pi}}\, Q \begin{pmatrix} j & 2 & j \\ -j & 0 & j \end{pmatrix}^{-1}
\end{aligned}$$

$$= (-1)^{j-m}\, Q \begin{pmatrix} j & j & 2 \\ m & -m & 0 \end{pmatrix} \begin{pmatrix} j & j & 2 \\ j & -j & 0 \end{pmatrix}^{-1}.$$

Using Table 5.1 for the $3J$ coefficients and simplifying gives

$$\sqrt{\frac{16\pi}{5}}\, \langle jm|Q_{20}|jm\rangle = \frac{3m^2 - j(j+1)}{j(2j-1)}\, Q.$$

5.9 Use Eq. (5.49), the result proved in Exercise 5.8, and the symmetries of the $6J$ coefficients. When I did it, I got a couple of extraneous square root factors. I'll have to check whether I made a mistake in the derivation, or whether the result that I asked you to derive is consistent with the definitions we have adopted.

5.11 Applying Eq. (5.56),

$$\begin{aligned} \langle l_1 l_2 L \| \boldsymbol{C}_k(1) \cdot \boldsymbol{C}_k(2) \| l_1' l_2' L' \rangle &= (-1)^{l_1' + l_2 + L} \sqrt{2L+1}\, \delta_{LL'} \\ &\quad \times \begin{Bmatrix} l_1 & l_1' & k \\ l_2' & l_2 & L \end{Bmatrix} \langle l_1 \| \boldsymbol{C}_k(1) \| l_1' \rangle \langle l_2 \| \boldsymbol{C}_k(2) \| l_2' \rangle \\ &= (-1)^{l_1' + l_2 + L} \sqrt{2L+1}\ \delta_{LL'}\ \sqrt{2l_1' + 1} \sqrt{2l_2' + 1} \\ &\quad \times \langle k0l_1'0|l_10\rangle \langle k0l_2'0|l_20\rangle \begin{Bmatrix} l_1 & l_1' & k \\ l_2' & l_2 & L \end{Bmatrix}, \end{aligned}$$

where we have used Eq. (5.34) and that $\boldsymbol{C}_k = \sqrt{4\pi/(2k+1)}\, \boldsymbol{Y}_k$ to write

$$\langle l \| \boldsymbol{C}_k \| l' \rangle = \sqrt{2l'+1}\, \langle k0l'0|l0\rangle.$$