**Chapter 4: Solutions to Odd-Numbered Exercises**

**4.1** Since there is only one way to couple $|jm\rangle$ and $|00\rangle$ to get $|j'm'\rangle$, we have

$$\langle jm00|j'm'\rangle = \delta_{jj'}\delta_{mm'}.$$

Then, by the symmetries of the Clebsch–Gordan coefficients under rearrangement of indices (see Eq. (4.76)),

$$\langle jmj' - m'|00\rangle = \frac{(-1)^{j-m}}{\sqrt{2j+1}}\,\delta_{jj'}\delta_{mm'}.$$

**4.3** Angular momentum 1 and $\frac{1}{2}$ can couple to give angular momentum $\frac{3}{2}$ and $\frac{1}{2}$. Since there is only one way to couple to $J = \frac{3}{2}, M = \frac{3}{2}$, the wavefunction $|JM\rangle = |jm\rangle|j'm'\rangle$ and corresponding Clebsch–Gordan coefficient $\langle j_1m_1j_2m_2|JM\rangle$ for that case must be,

$$|\tfrac{3}{2}\tfrac{3}{2}\rangle = |11\rangle|\tfrac{1}{2}\tfrac{1}{2}\rangle \qquad \langle 11\tfrac{1}{2}\tfrac{1}{2}|\tfrac{3}{2}\tfrac{3}{2}\rangle = 1.$$

Operate on both sides of the wavefunction equation with $J_- = j_- + j'_-$, with

$$\begin{aligned}
J_-|JM\rangle &= \sqrt{J(J+1) - M(M-1)}\,|J, M-1\rangle, \\
j_-|jm\rangle &= \sqrt{j(j+1) - m(m-1)}\,|j, m-1\rangle, \\
j'_-|j'm'\rangle &= \sqrt{j'(j'+1) - m'(m'-1)}\,|j', m'-1\rangle,
\end{aligned}$$

to find the result

$$|\tfrac{3}{2}\tfrac{1}{2}\rangle = \left(\sqrt{\tfrac{2}{3}}\,|10\rangle|\tfrac{1}{2}\tfrac{1}{2}\rangle + \tfrac{1}{\sqrt{3}}\,|11\rangle|\tfrac{1}{2} - \tfrac{1}{2}\rangle\right).$$

Multiply this from the left by $\langle 10\frac{1}{2}\frac{1}{2}| \equiv \langle 10|\langle\frac{1}{2}\frac{1}{2}|$ and $\langle 11|\langle\frac{1}{2} - \frac{1}{2}|$, respectively, to give the Clebsch–Gordan coefficients

$$\langle 10\tfrac{1}{2}\tfrac{1}{2}|\tfrac{3}{2}\tfrac{1}{2}\rangle = \sqrt{\tfrac{2}{3}} \qquad \langle 11\tfrac{1}{2} - \tfrac{1}{2}|\tfrac{3}{2}\tfrac{1}{2}\rangle = \tfrac{1}{\sqrt{3}}.$$

Now operate with $J_- = j_- + j'_-$ on both sides of

$$|\tfrac{3}{2}\tfrac{1}{2}\rangle = \left(\sqrt{\tfrac{2}{3}}\,|10\rangle|\tfrac{1}{2}\tfrac{1}{2}\rangle + \tfrac{1}{\sqrt{3}}\,|11\rangle|\tfrac{1}{2} - \tfrac{1}{2}\rangle\right)$$

to obtain

$$|\tfrac{3}{2} - \tfrac{1}{2}\rangle = \tfrac{1}{\sqrt{3}}\,|1-1\rangle|\tfrac{1}{2}\tfrac{1}{2}\rangle + \tfrac{1}{\sqrt{6}}\,|10\rangle|\tfrac{1}{2} - \tfrac{1}{2}\rangle + \tfrac{1}{\sqrt{6}}\,|10\rangle|\tfrac{1}{2} - \tfrac{1}{2}\rangle.$$

Multiplying this from the left by $\langle 1\,-1|\langle \frac{1}{2}\frac{1}{2}|$ and $\langle 10|\langle \frac{1}{2}\,-\frac{1}{2}|$, respectively, gives the Clebsch–Gordan coefficients

$$\langle 1\,-1\tfrac{1}{2}\tfrac{1}{2}|\tfrac{3}{2}\,-\tfrac{1}{2}\rangle = \tfrac{1}{\sqrt{3}} \qquad \langle 10\tfrac{1}{2}\,-\tfrac{1}{2}|\tfrac{3}{2}\,-\tfrac{1}{2}\rangle = \tfrac{1}{\sqrt{6}} + \tfrac{1}{\sqrt{6}} = \sqrt{\tfrac{2}{3}}.$$

Other Clebsch–Gordan coefficients may be obtained in a similar way with proper care taken to ensure that states are orthogonal.

**4.5** From Eq. (4.77) and the column permutation symmetries for the $3J$ coefficients summarized in §4.3.8,

$$\langle j_1 m_1 j_2 m_2 | JM\rangle = (-1)^{J-M-2j_2}\sqrt{2J+1}\begin{pmatrix} J & j_2 & j_1 \\ -M & m_2 & m_1 \end{pmatrix}.$$

This may be used with the formulas in Table 4.4 to determine the required Clebsch–Gordan coefficients. For example, from the preceding equation,

$$\langle 2011|11\rangle = \sqrt{3}\begin{pmatrix} 1 & 1 & 2 \\ -1 & 1 & 0 \end{pmatrix}$$

and from Table 4.4

$$\begin{pmatrix} a & a & 2 \\ b & -b & 0 \end{pmatrix} = (-1)^{a-b}\frac{3b^2 - a(a+1)}{[a(a+1)(2a+3)(2a+1)(2a-1)]^{1/2}},$$

so setting $a = 1$ and $b = -1$, we find that

$$\langle 2011|11\rangle = \sqrt{3}\begin{pmatrix} 1 & 1 & 2 \\ -1 & 1 & 0 \end{pmatrix} = \sqrt{3}\times\sqrt{\frac{1}{30}} = \frac{1}{\sqrt{10}}.$$

By a similar procedure, we find from Table 4.4 that

$$\langle 2110|11\rangle = \sqrt{3}\begin{pmatrix} 1 & 1 & 2 \\ -1 & 0 & 1 \end{pmatrix} = \sqrt{3}\times\frac{-1}{\sqrt{10}} = -\sqrt{\frac{3}{10}}$$

and that

$$\langle 221\,-1|11\rangle = \sqrt{3}\begin{pmatrix} 1 & 1 & 2 \\ -1 & -1 & 2 \end{pmatrix} = \sqrt{3}\times\frac{1}{\sqrt{5}} = \sqrt{\frac{3}{5}}.$$