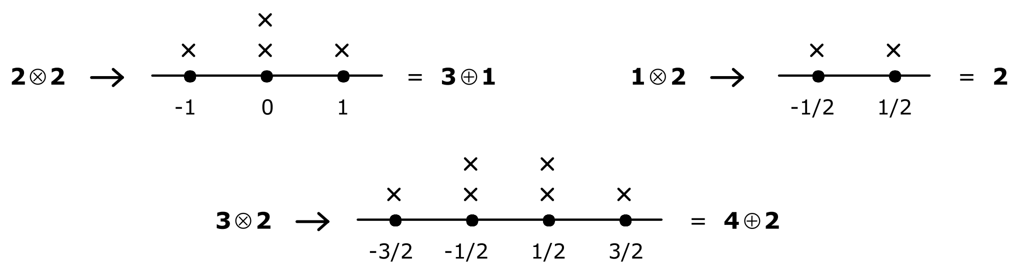
Chapter 2: Solutions to Odd-Numbered Exercises

2.1 Do it in two steps. First use the highest-weight algorithm to show that $\mathbf{2} \otimes \mathbf{2} = \mathbf{3} \oplus \mathbf{1}$. Then take the product of that with the remaining $\mathbf{2}$ to show that

$$\mathbf{2} \otimes \mathbf{2} \otimes \mathbf{2} = (\mathbf{3} \oplus \mathbf{1}) \otimes \mathbf{2} = \mathbf{4} \oplus \mathbf{2} \oplus \mathbf{2}.$$

The relevant highest-weight decompositions are illustrated in the following diagrams:



(see the example discussed in §2.3.5).

2.3 Evaluate the quantitity $U(\alpha)^{-1}U(\beta)^{-1}U(\alpha)U(\beta)$ for two group elements

$$U(\alpha) = e^{i(\alpha_1 X_1 + \alpha_2 X_2, + \ldots)} \qquad U(\beta) = e^{i(\beta_1 X_1 + \beta_2 X_2, + \ldots)}$$

lying near the origin $U(0)$ in the parameter space, where $\alpha = (\alpha_1, \alpha_2, \ldots)$ and $\beta = (\beta_1, \beta_2, \ldots)$ are two sets of angles parameterizing nearby points and $X = (X_1, X_2, \ldots)$ are the group generators. (This quantity measures the commutivity of the group because if the group elements commute,

$$U(\alpha)^{-1}U(\beta)^{-1}U(\alpha)U(\beta) = U(\alpha)^{-1}U(\beta)^{-1}U(\beta)U(\alpha) = U(\alpha)^{-1}U(\alpha) = 1,$$

and deviations of the product from unity indicate a failure to commute.) The product defined above must give a new element of the group, say $U(\gamma)$,

$$U(\alpha)^{-1}U(\beta)^{-1}U(\alpha)U(\beta) = U(\gamma).$$

Substituting the exponential forms for all group elements and expanding in the vicinity of the origin for the exponentials on both sides of the equation gives

$$1 + \sum_{k\ell} \alpha_k \beta_\ell (X_k X_\ell - X_\ell X_k) + \ldots = 1 + \sum_m \gamma_m X_m + \ldots$$

Comparing term by term, consistency then requires that

$$[X_k, X_\ell] = \sum_m c^m_{k\ell} X_m,$$

where $[X_k, X_\ell] \equiv X_k X_\ell - X_\ell X_k$ and $c^m_{k\ell}$ is an expansion coefficient.

2.5 The adjoint representation is generated by the structure constants; in particular, for $SU(2)$ the adjoint representation is given in terms of three 3×3 matrices t_a, with

$$(t_a)_{ij} = -i\varepsilon_{ija} \qquad \varepsilon_{ija} = \begin{cases} +1 & \text{if } ijk \text{ cyclic} \\ \text{-1} & \text{if } ijk \text{ anticyclic} \\ 0 & \text{otherwise} \end{cases}$$

For example, if $a = 1$,

$$(t_1)_{23} = -i\varepsilon_{231} = -i \qquad (t_1)_{32} = -i\varepsilon_{321} = i,$$

and all other components $(t_1)_{ij}$ are zero. Then,

$$t_1 = -i \begin{pmatrix} \varepsilon_{111} & \varepsilon_{121} & \varepsilon_{131} \\ \varepsilon_{211} & \varepsilon_{221} & \varepsilon_{231} \\ \varepsilon_{311} & \varepsilon_{321} & \varepsilon_{331} \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}$$

By a similar construction,

$$t_2 = \begin{pmatrix} 0 & 0 & i \\ 0 & 0 & 0 \\ -i & 0 & 0 \end{pmatrix} \qquad t_3 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Since this is a rank-1 algebra, we may diagonalize one generator. Let's choose t_3,

$$\begin{vmatrix} -\lambda & -i & 0 \\ i & -\lambda & 0 \\ 0 & 0 & -\lambda \end{vmatrix} = 0.$$

The characteristics λ then satisfy $\lambda^3 + \lambda = 0$, which has solutions $\lambda = 0, \pm 1$. Therefore, we may choose the diagonal form

$$t_3' = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

To transform t_1 and t_2 to this new basis in which t_3 is diagonal, we need the matrix c that satisfies

$$c^{-1} t_3 c = t_3'.$$

This matrix is formed from columns of eigenvectors of t'_3 and a suitable choice for c and its inverse are

$$c = \begin{pmatrix} \frac{-i}{\sqrt{2}} & 0 & \frac{i}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ 0 & 1 & 0 \end{pmatrix} \qquad c^{-1} = \begin{pmatrix} \frac{i}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \\ \frac{-i}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \end{pmatrix}$$

Then it is easily verified by explicit matrix multiplication that

$$t'_1 = c^{-1} t_1 c = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & -i & 0 \\ i & 0 & i \\ 0 & -i & 0 \end{pmatrix}$$

$$t'_2 = c^{-1} t_2 c = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & -1 & 0 \\ -1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

$$t'_3 = c^{-1} t_3 c = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

constitutes a new basis in which t_3 is diagonal.

2.7 For a concise but clear overview of the quasispin model, see Ring and Schuck, *The Nuclear Many-Body Problem*, §6.2. This solution follows that presentation.

(a) The commutators may be worked out explicitly. For example

$$\begin{aligned}
[s_+^{(m)}, s_-^{(m)}] &= a_m^\dagger a_{-m}^\dagger a_{-m} a_m - a_{-m} a_m a_m^\dagger a_{-m}^\dagger \\
&= a_m^\dagger a_{-m}^\dagger a_{-m} a_m - a_{-m}(1 - a_m^\dagger a_m) a_{-m}^\dagger \\
&= a_m^\dagger a_{-m}^\dagger a_{-m} a_m - a_{-m} a_{-m}^\dagger - a_{-m} a_m^\dagger a_m a_{-m}^\dagger \\
&= a_m^\dagger a_{-m}^\dagger a_{-m} a_m - 1 + a_{-m}^\dagger a_{-m} - a_m^\dagger a_{-m} a_m a_{-m}^\dagger \\
&= a_m^\dagger a_{-m}^\dagger a_{-m} a_m - 1 + a_{-m}^\dagger a_{-m} - a_m^\dagger a_{-m} a_{-m}^\dagger a_m \\
&= a_m^\dagger a_{-m}^\dagger a_{-m} a_m - 1 + a_{-m}^\dagger a_{-m} + a_m^\dagger (1 - a_{-m}^\dagger a_{-m}) a_m \\
&= a_m^\dagger a_m + a_{-m}^\dagger a_m - 1 \\
&= 2 s_0^{(m)}.
\end{aligned}$$

Calculating the other commutators in a similar manner, we obtain

$$[s_+^{(m)}, s_-^{(m)}] = 2s_0^{(m)} \qquad [s_0^{(m)}, s_\pm^{(m)}] = \pm s_\pm^{(m)},$$

which is an $SU(2)$ algebra with s playing the role of a "quasispin". Since

$$s_0^{(m)} = \tfrac{1}{2}(a_m^\dagger a_m + a_{-m}^\dagger a_{-m} - 1),$$

it has the following eigenvalues:

1. $s_0^{(m)} = \frac{1}{2}$ if the $\pm m$ level contains a pair.
2. $s_0^{(m)} = -\frac{1}{2}$ if the $\pm m$ level is empty.
3. $s_0^{(m)} = 0$ if there is one particle in the $\pm m$ level.

Therefore, for each level m we may define a quasispin vector $\boldsymbol{s}^{(m)}$ and a total quasispin $\boldsymbol{S}$ for the system,

$$\boldsymbol{S} = \sum_{m>0} \boldsymbol{s}^{(m)}.$$

The total quasispin $\boldsymbol{S}$ must also be algebraically an "angular momentum" (quasispin) and is a generator of $SU(2)$.

(b) Defining the operators

$$S_+ \equiv \sum_{m>0} a_m^\dagger a_{-m}^\dagger = \sum_{m>0} s_+^{(m)}$$

$$S_- \equiv \sum_{m>0} a_m a_{-m} = \sum_{m>0} s_-^{(m)}$$

permits the Hamiltonian to be expressed as

$$\begin{aligned} H &= -G \sum_{m,m'>0} a_{m'}^\dagger a_{-m'}^\dagger a_{-m} a_m \\ &= -G S_+ S_- \\ &= -G \sum_{mm'} s_+^{(m')} s_-^{(m)}. \end{aligned}$$

But $\boldsymbol{S}$, and $S_\pm$ are generators of $SU(2)$, so they must obey the relation familiar from angular momentum,

$$S^2 = \tfrac{1}{2}(S_+S_- + S_-S_+) + S_0^2,$$

where S^2 is the total quasispin Casimir operator and

$$S_0 = \tfrac{1}{2}\sum_{m>0}(a_m^\dagger a_m + a_{-m}^\dagger a_{-m} - 1) = \sum_{m>0} s_0^{(m)}.$$

Therefore,

$$\begin{aligned} S_+S_- &= 2S^2 - 2S_0^2 - S_-S_+ \\ &= S^2 - S_0^2 + S_0, \end{aligned}$$

where $[S_+, S_-] = 2S_0$ was used in the second step. This permits us to express the Hamiltonian entirely in terms of diagonal quasispin operators:

$$H = -GS_+S_- = -G(S^2 - S_0^2 + S_0).$$

Choosing a basis where S^2 and S_0 (and hence H) are diagonal,

$$\langle SS_0|S^2|SS_0\rangle = S(S+1).$$

Furthermore, the operator S_0 may be written as

$$S_0 = \tfrac{1}{2}\sum_{m>0}(a_m^\dagger a_m + a_{-m}^\dagger a_{-m}) - \tfrac{1}{2}\sum_{m>0} 1.$$

The first term just counts the number of pairs of particles N in $\pm m$ orbitals in the shell, while the second term counts the total pair degeneracy Ω of the shell (twice the maximum number of particles it could hold). Therefore,

$$\langle SS_0|S_0|SS_0\rangle = \tfrac{1}{2}(N - \Omega)$$

and the energy is given by

$$E = \langle SS_0|H|SS_0\rangle = -G\left(S(S+1) - \tfrac{1}{4}(N-\Omega)^2 + \tfrac{1}{2}(N-\Omega)\right).$$

(c) Defining the *seniority* ν through

$$S = \tfrac{1}{2}(\Omega - \nu),$$

permits the energy to be expressed as

$$E(N,\nu) = -\frac{G}{4}\left(\nu^2 - 2\nu(\Omega+1) + 2N(\Omega+1) - N^2\right).$$

Successive quasispin multiplets differ in S by 1, so the the difference in energy between adjacent multiplets is

$$\begin{aligned} \Delta E &= E(S-1) - E(S) \\ &= -G\left((S-1)(S-1+1) - S(S+1)\right) \\ &= -G(S^2 - S - S^2 - S) \\ &= 2GS \end{aligned}$$

and the spectrum is independent of the number of particles in the shell. Since $-GS(S+1)$ is an attractive contribution to the Hamiltonian (assuming $G > 0$), the ground state results from the full alignment of quasispins. Because

$$\boldsymbol{S} = \sum_{m>0} \boldsymbol{s}^{(m)}$$

and the length of each $\boldsymbol{s}^{(m)}$ is $\frac{1}{2}$, the ground-state quasispin S_{g} is

$$S_{\mathrm{g}} = \tfrac{1}{2}\Omega.$$

The ground state is a seniority-zero state since

$$\nu_{\mathrm{g}} = \Omega - 2S_{\mathrm{g}} = \Omega - 2\,\frac{\Omega}{2} = 0.$$

The first excited state will correspond to the second largest alignment of quasispins,

$$S_1 = \tfrac{1}{2}\Omega - 1,$$

and is a seniority-two state:

$$\nu_1 = \Omega - 2S_1 = \Omega - 2\,\frac{\Omega}{2} + 2 = 2.$$

(d) For two identical particles in the j-shell, the total angular momentum J can take the values $J = 0, 2, 4, \ldots 2j-1$, where the Pauli principle forbids odd spins for the pair. (Note: this is now the real angular momentum, not the quasispin!) For even number of particles the seniorities can take the values $\nu = 0, 2, \ldots, N$, so only the $\nu = 0, 2$ states can occur for two particles in the j-shell. Examining the Hamiltonian, we see that the energy is zero, *unless the two particles are paired to zero angular momentum,* and less than zero if $J = 0$. That is, $a^\dagger_{m'} a^\dagger_{-m'} a_{-m} a_m$ contributes only for a two-particle state in which the m and m' have opposite signs, meaning that the total angular momentum of the state is zero. Therefore, for $G > 0$ (attractive pairing interaction), the spectrum is

$$\begin{array}{rcl} (E=0)\ \nu = 2 & \underline{\hspace{4cm}} & J = 2, 4, \ldots, 2j-1 \\ (E=-G\Omega)\ \nu = 0 & \underline{\hspace{4cm}} & J = 0 \end{array}$$

The seniority labeling in this diagram results from the previous argument that the ground state must be $\nu = 0$ and that only $\nu = 0, 2$ states are allowed for a single pair of particles. Generally, in this model the seniority ν measures the number of pairs not coupled to total angular momentum $J = 0$. Notice that the energy gap associated with pair breaking,

$$\Delta E = E(N, \nu = 2) - E(N, \nu = 0) = G\Omega,$$

depends only on the shell degeneracy and the coupling strength G; it is independent of the particle number.

(e) The ground-state binding energy is

$$\begin{aligned} B_{\mathrm{g}} &= -E(N, \nu = 0) \\ &= \frac{G}{4}\left(2N(\Omega + 1) - N^2\right) \\ &= \frac{1}{2}G\Omega N\left(1 - \frac{N-2}{2\Omega}\right) \\ &\simeq \frac{G\Omega}{2}N \qquad (N << \Omega). \end{aligned}$$

This spectrum, approximately linear in the pair number for small shell degeneracy, is termed a *pair vibrational spectrum.*

Quasispin models are of physical relevance for any case in which there are fermion shells with a strong tendency for pairs of particles to couple to zero total angular momentum. Such models, and their generalizations, have found substantial application in nuclear, atomic, and condensed matter physics.

2.9 Multiplying $[T_a, T_b] = if_{abc}T_c$ from the right by T_d and taking the trace of both sides of the equation gives

$$\mathrm{Tr}\,([T_a, T_b]T_d) = if_{abc}\mathrm{Tr}\,(T_cT_d).$$

But the generators are normalized $\mathrm{Tr}\,(T_aT_b) = \lambda\delta_{ab}$, so we may write

$$\begin{aligned} f_{abc} &= -i\lambda^{-1}\mathrm{Tr}\,([T_a, T_b]T_c) \\ &= -i\lambda^{-1}\mathrm{Tr}\,(T_aT_bT_c - T_bT_aT_c) \\ &= -i\lambda^{-1}\mathrm{Tr}\,(T_bT_cT_a - T_cT_bT_a) \\ &= -i\lambda^{-1}\mathrm{Tr}\,([T_b, T_c]T_a) \\ &= f_{bca}, \end{aligned}$$

where in line three the cyclic property of the trace has been used. Likewise, by a different cyclic permutation in line three, $f_{abc} = f_{cab}$. Therefore, $f_{abc} = f_{bca} = f_{cab}$.

Finally, $[A, B] = -[B, A]$ implies that $f_{abc} = -f_{bac}$, so combining all these results gives

$$f_{abc} = f_{bca} = f_{cab} = -f_{bac} = -f_{cba} = -f_{acb}$$

for the symmetries under exchange of indices for the structure constants f_{abc} of a compact, semisimple algebra.