Chapter 1: Solutions to Odd-Numbered Exercises

1.1 Define the permutations

$$(\ ) = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix} \qquad (12) = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix} \qquad (23) = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix}$$

$$(13) = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix} \qquad (123) = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix} \qquad (321) = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}$$

so that, for example, $(12)\{abc\} \to \{bac\}$. Carrying out all possible combinations, the multiplication table for $A \cdot B$ is

$B\backslash A$	e	(12)	(23)	(13)	(123)	(321)
e	e	(12)	(23)	(13)	(123)	(321)
(12)	(12)	e	(123)	(321)	(23)	(13)
(23)	(23)	(321)	e	(123)	(13)	(12)
(13)	(13)	(123)	(321)	e	(12)	(23)
(123)	(123)	(13)	(12)	(23)	(321)	e
(321)	(321)	(23)	(13)	(12)	e	(123)

where e is the identity, $e \equiv (\ )$. From the table this is a group, because all postulates are satisfied. The identity and any two of the transpositions of two objects are closed under multiplication, so these form 2-element subgroups that are isomorphic to S_2. The set of even permutations $A_3 = \{e, (123), (321)\}$ is closed under multiplication, so this too is a subgroup. The multiplication table for C_3 can be deduced from geometry. The operators 1, c_3, c_3^2, c_3^3, and c_3^4 rotate by 0, $\frac{2\pi}{3}$, $\frac{4\pi}{3}$, $2\pi = 0$, and $\frac{2\pi}{3}$, respectively, so the multiplication table is

C_3	e	c_3	c_3^2
e	e	c_3	c_3^2
c_3	c_3	c_3^2	e
c_3^2	c_3^2	e	c_3

$\Longleftrightarrow$

A_3	e	(123)	(321)
e	e	(123)	(321)
(123)	(123)	(321)	e
(321)	(321)	e	(123)

and we have the correspondence

$$e \longleftrightarrow e \qquad c_3 \longleftrightarrow (123) \qquad c_3^2 \longleftrightarrow (321).$$

1.3 An invariant subgroup consists of whole classes. For S_3 there are three classes: $\{e\}$, $\{(12),(23),(13)\}$, and $\{(123),(321)\}$, as may be verified by making similarity transforms (see Eq. (1.34)). Therefore, $\{e,(123),(321)\}$ (the *alternating subgroup*) is invariant because it consists of two whole classes. The subgroup $\{e,(12)\}$ is not invariant because e is a whole class but (12) is only part of a class.

1.5 (a) $a \sim a$ requires that there be some group element g such that $a = gag^{-1}$. The choice $g = e$ obviously satisfies this. (b) If $a \sim b$, then $a = gbg^{-1}$ where $g \in G$. Multiply from the left by g^{-1} and from the right by g to give $g^{-1}ag = b$, which implies that $b \sim a$. (c) If $a \sim b$ and $b \sim c$, then there are elements $p, q \in G$ such that

$$a = pbp^{-1} \qquad b = qcq^{-1}.$$

Therefore,

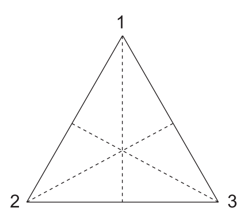
$$a = pbp^{-1} = pqcq^{-1}p^{-1} = pqc(pq)^{-1},$$

where we have used that the reciprocal of a matrix product is formed by taking reciprocals of the factors in reverse order:

$$(ABCD)^{-1} = D^{-1}C^{-1}B^{-1}A^{-1}.$$

Then, since pq must be a group element, $a \sim c$.

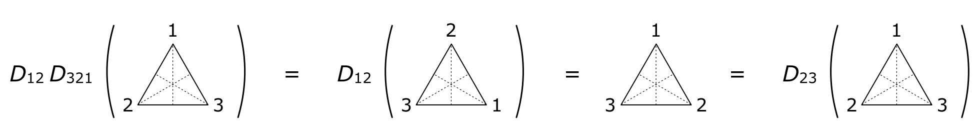
1.7 From the geometry of the figure



a multiplication table may be constructed by applying two successive operations. Let

$$\begin{aligned} D_{12} &= \text{reflection about 3-axis} & e &= \text{identity} \\ D_{13} &= \text{reflection about 2-axis} & D_{321} &= \text{clockwise rotation by } 2\pi/3 \\ D_{23} &= \text{reflection about 1-axis} & D_{123} &= \text{clockwise rotation by } 4\pi/3. \end{aligned}$$

Then, for example,



and $D_{12}D_{321} = D_{23}$. By carrying out all such pairs of operations, a 6×6 multiplication table may be constructed that is closed (group property) and equivalent to the table of Exercise 1.1, with the identification $D_{pq} \to (pq)$. Thus the group is isomorphic to S_3. We have already seen in Exercise 1.1 that S_3 has four proper, distinct subgroups: the alternating group, and three S_2 subgroups formed from the identity and one of the three independent transpositions.

1.9 Consider an operator $U(a)$ that rotates by an infinitesimal amount a around a specified axis. Applying to a wavefunction $\psi(\theta)$,

$$U(a)\psi(\theta) = \psi(\theta + a).$$

Expand in a Taylor series about θ:

$$\begin{aligned}
\psi(\theta + a) &= \psi(\theta) + a\frac{d\psi(\theta)}{d\theta} + \frac{a^2}{2!}\frac{d^2\psi(\theta)}{d\theta^2} + \dots \\
&= \sum_{n=0}^{\infty} \frac{a^n}{n!}\frac{d^n}{d\theta^n}\psi(\theta) = e^{a(d/d\theta)}\psi(\theta) \\
&= e^{ia(-id/d\theta)}\psi(\theta) = e^{\frac{i}{\hbar}aL}\psi(\theta),
\end{aligned}$$

where $L \equiv \frac{\hbar}{i} d/d\theta$ is the generator of infinitesimal rotations about this axis,

$$U(a) \simeq 1 + \frac{i}{\hbar}aL.$$

But since the rotations form a continuous and analytical group, all finite rotations connected continuously to the identity can be generated by successive applications of the infinitesimal generator L.

1.11 (a) We have from Eqns. (1.48)–(1.50),

$$A^{(\alpha\times\beta)}_{ij,k\ell}(G_a) \equiv A^{(\alpha)}_{ik}(G_a)A^{(\beta)}_{j\ell}(G_a),$$

for the elements of a direct product matrix. Then under regular matrix multiplication of two such matrices that are themselves direct products, the elements are

$$\begin{aligned}
[A^{(\alpha\times\beta)}(G_a)A^{(\alpha\times\beta)}(G_b)]_{ij,k\ell} &= \sum_{mn} A^{(\alpha\times\beta)}_{ij,mn}(G_a)A^{(\alpha\times\beta)}_{mn,k\ell}(G_b) \\
&= \sum_{mn} A^{(\alpha)}_{im}(G_a)A^{(\beta)}_{jn}(G_a)A^{(\alpha)}_{mk}(G_b)A^{(\beta)}_{n\ell}(G_b) \\
&= \sum_{m} A^{(\alpha)}_{im}(G_a)A^{(\alpha)}_{mk}(G_b)\sum_{n} A^{(\beta)}_{jn}(G_a)A^{(\beta)}_{n\ell}(G_b) \\
&= A^{(\alpha)}_{ik}(G_aG_b)A^{(\beta)}_{j\ell}(G_aG_b) \\
&= A^{(\alpha\times\beta)}_{ij,k\ell}(G_aG_b),
\end{aligned}$$

where we have used Eq. (1.13) for the original matrices, $T(G_a) \cdot T(G_b) = T(G_a G_b)$. Therefore, the representation condition (1.13) is fulfilled for direct product matrices: if A and B are representations, then their direct product $A \otimes B$ is also a representation.

(b) For the characters,

$$
\begin{aligned}
\chi^{(\alpha\times\beta)}(G_a) &= \sum_{ij} A^{(\alpha\times\beta)}_{ij,ij}(G_a) \\
&= \sum_{ij} A^{(\alpha)}_{ii}(G_a) T^{(\beta)}_{jj}(G_a) \\
&= \chi^{(\alpha)}(G_a)\chi^{(\beta)}(G_a).
\end{aligned}
$$

Therefore, the character of a direct product of two representations is the product of characters for the two representations.

1.13 Define coset multiplication by $pHqH = pqH$. Then,

(a) For H an invariant subgroup, this is a consistent definition and leads to *closure* because $Hg = gH$ and

$$pHqH = pHHq = pHq = pqH,$$

where we have used that under this multiplication law $eHeH = eeH = H$, so $HH = H$. Therefore, the product of two cosets under coset multiplication is itself a coset, since if $p \in G$ and $q \in G$, then $pq \in G$.

(b) The coset $E \equiv eH = H$ acts as an *identity* since, by the coset multiplication law,

$$eHqH = eqH = qH.$$

(c) For each coset pH there is a *unique inverse* $p^{-1}H$, since

$$(pH)(p^{-1}H) = (pp^{-1})H = eH = E.$$

(d) Finally, coset multiplication is *associative* because the original group multiplication is associative:

$$
\begin{aligned}
(pH)\left((qH)(rH)\right) &= (pH)(qrHH) = p(qr)HHH \\
&= (pq)rHHH = \left((pH)(qH)\right)(rH).
\end{aligned}
$$

Therefore, the cosets form a group under the coset multiplication law if H is an invariant subgroup (note that the invariance of H in G is essential to this conclusion).