Supplement: Emergent Fermion Dynamical Symmetries for Monolayer Graphene in a Strong Magnetic Field

Fletcher Williams, Lianao Wu, and Mike Guidry

This document provides supplemental material and proofs of some important equations for the review article "Emergent Fermion Dynamical Symmetries for Monolayer Graphene in a Strong Magnetic Field" by Mike Guidry, Lianao Wu, and Fletcher Williams.

1. Introduction

In the following all citations of sections, subsections, equation numbers, figure numbers, and table numbers are by default references to the primary document "Emergent Dynamical Symmetries for Monolayer Graphene in a Strong Magnetic Field". If a reference is flagged by "[this document]", it is instead a reference to objects in the present Supplement document.

2. Pairing operators

As a representative example, from Eq. (126) the $S=1, M_S=0, T=0$ pair is,

$$A_{00}^{\dagger 10} = \sum_{m_k} \sum_{m_1 m_2} \sum_{n_1 n_2} \langle \tfrac12\, m_1\, \tfrac12\, m_2 | 1\, 0 \rangle \langle \tfrac12\, n_1\, \tfrac12\, n_2 | 0\, 0 \rangle c^\dagger_{m_1 n_1 m_k} c^\dagger_{m_2 n_2 -m_k}. \tag{1}$$

But generally for Clebsch–Gordan coefficients,

$$\langle \tfrac12\, n_1\, \tfrac12\, n_2 | 0\, 0 \rangle = \frac{(-1)^{1/2-n_1}}{\sqrt{2}}\, \delta_{n_1,-n_2}, \tag{2}$$

so that

$$A_{00}^{\dagger 10} = \sum_{m_k} \sum_{m_1 m_2} \sum_{n_1} \frac{(-1)^{1/2-n_1}}{\sqrt{2}} \langle \tfrac12\, m_1\, \tfrac12\, m_2 | 1\, 0 \rangle c^\dagger_{m_1 n_1 m_k} c^\dagger_{m_2 -n_1 -m_k}.$$

The remaining Clebsch–Gordan coefficient in this expression vanishes unless $m_2 = -m_1$, so

$$A_{00}^{\dagger 10} = \sum_{m_k} \sum_{m_1 n_1} \frac{(-1)^{1/2-n_1}}{\sqrt{2}} \langle \tfrac12\, m_1\, \tfrac12\, -m_1 | 1\, 0 \rangle c^\dagger_{m_1 n_1 m_k} c^\dagger_{-m_1 -n_1 -m_k}.$$

Writing the four terms in the sum over m_1 and n_1 out explicitly for $n_1 = (-\frac12, \frac12)$ and $m_1 = (-\frac12, \frac12)$ gives

$$\begin{aligned} A_{00}^{\dagger 10} = &-\frac{1}{\sqrt{2}} \sum_{m_k} \langle \tfrac12\, -\tfrac12\, \tfrac12\, \tfrac12 | 1\, 0 \rangle c^\dagger_{-\frac12 -\frac12 m_k} c^\dagger_{\frac12 \frac12 -m_k} \\ &+\frac{1}{\sqrt{2}} \sum_{m_k} \langle \tfrac12\, -\tfrac12\, \tfrac12\, \tfrac12 | 1\, 0 \rangle c^\dagger_{-\frac12 \frac12 m_k} c^\dagger_{\frac12 -\frac12 -m_k} \\ &-\frac{1}{\sqrt{2}} \sum_{m_k} \langle \tfrac12\, \tfrac12\, \tfrac12\, -\tfrac12 | 1\, 0 \rangle c^\dagger_{\frac12 -\frac12 m_k} c^\dagger_{-\frac12 \frac12 -m_k} \\ &+\frac{1}{\sqrt{2}} \sum_{m_k} \langle \tfrac12\, \tfrac12\, \tfrac12\, -\tfrac12 | 1\, 0 \rangle c^\dagger_{\frac12 \frac12 m_k} c^\dagger_{-\frac12 -\frac12 -m_k}. \end{aligned}$$

Utilizing from Table I [this document] that

$$\langle \tfrac12\, -\tfrac12\, \tfrac12\, \tfrac12 | 1\, 0 \rangle = \langle \tfrac12\, \tfrac12\, \tfrac12\, -\tfrac12 | 1\, 0 \rangle = \frac{1}{\sqrt{2}}$$

we may write

$$
\begin{aligned}
A_{00}^{\dagger 10} &= -\frac{1}{2}\sum_{m_k} c^{\dagger}_{-\frac{1}{2}-\frac{1}{2}m_k} c^{\dagger}_{\frac{1}{2}\frac{1}{2}-m_k} + \frac{1}{2}\sum_{m_k} c^{\dagger}_{-\frac{1}{2}\frac{1}{2}m_k} c^{\dagger}_{\frac{1}{2}-\frac{1}{2}-m_k} \\
&\quad -\frac{1}{2}\sum_{m_k} c^{\dagger}_{\frac{1}{2}-\frac{1}{2}m_k} c^{\dagger}_{-\frac{1}{2}\frac{1}{2}-m_k} + \frac{1}{2}\sum_{m_k} c^{\dagger}_{\frac{1}{2}\frac{1}{2}m_k} c^{\dagger}_{-\frac{1}{2}-\frac{1}{2}-m_k} \\
&= -\frac{1}{2}\sum_{m_k} c^{\dagger}_{4m_k} c^{\dagger}_{1-m_k} + \frac{1}{2}\sum_{m_k} c^{\dagger}_{2m_k} c^{\dagger}_{3-m_k} - \frac{1}{2}\sum_{m_k} c^{\dagger}_{3m_k} c^{\dagger}_{2-m_k} + \frac{1}{2}\sum_{m_k} c^{\dagger}_{1m_k} c^{\dagger}_{4-m_k} \\
&= \frac{1}{2}A^{\dagger}_{14} + \frac{1}{2}A^{\dagger}_{23} + \frac{1}{2}A^{\dagger}_{23} + \frac{1}{2}A^{\dagger}_{14} \\
&= A^{\dagger}_{14} + A^{\dagger}_{23},
\end{aligned} \tag{3}
$$

where in the second equation the mapping between spin and isospin quantum numbers and the label a in Fig. 25(a) has been used to replace labels (σ, τ) with the label a, we have performed manipulations such as

$$
-\frac{1}{2}\sum_{m_k} c^{\dagger}_{4m_k} c^{\dagger}_{1-m_k} = \frac{1}{2}\sum_{m_k} c^{\dagger}_{1-m_k} c^{\dagger}_{4m_k} = \frac{1}{2}\sum_{-m_k} c^{\dagger}_{1m_k} c^{\dagger}_{4-m_k} = \frac{1}{2}A^{\dagger}_{14},
$$

where the first equality is because independent fermion creation operators anticommute, the second equality is because m_k is a dummy summation index that may be replaced with another summation index, and the third equality employed the definition of $A^{\dagger}_{ab}$ in Eq. (119). The other five possibilities may be determined in a completely analogous way, with the results

$$
\begin{array}{lll}
A_{00}^{\dagger 10} = A^{\dagger}_{14} + A^{\dagger}_{23} & A_{10}^{\dagger 10} = \sqrt{2}A^{\dagger}_{13} & A_{-10}^{\dagger 10} = \sqrt{2}A^{\dagger}_{24}, \\
A_{00}^{\dagger 01} = A^{\dagger}_{14} - A^{\dagger}_{23} & A_{01}^{\dagger 01} = \sqrt{2}A^{\dagger}_{12} & A_{0-1}^{\dagger 01} = \sqrt{2}A^{\dagger}_{34}.
\end{array} \tag{4}
$$

The hermitian conjugates of these give the six corresponding pair annihilation operators in coupled representation. These are the generators given in Eq. (127), up to a normalization.

To take another example, from Eq. (126) the $S = 0, T = 1, M_T = 1$ pair is,

$$
A_{01}^{\dagger 01} = \sum_{m_k}\sum_{m_1 m_2}\sum_{n_1 n_2} \langle \tfrac{1}{2}\, m_1\, \tfrac{1}{2}\, m_2 | 0\, 0 \rangle \langle \tfrac{1}{2}\, n_1\, \tfrac{1}{2}\, n_2 | 1\, 1 \rangle c^{\dagger}_{m_1 n_1 m_k} c^{\dagger}_{m_2 n_2 -m_k}. \tag{5}
$$

But generally for Clebsch–Gordan coefficients,

$$
\langle \tfrac{1}{2}\, m_1\, \tfrac{1}{2}\, m_2 | 0\, 0 \rangle = \frac{(-1)^{1/2-m_1}}{\sqrt{2}}\, \delta_{m_1,-m_2},
$$

so that

$$
A_{01}^{\dagger 01} = \sum_{m_k}\sum_{n_1 n_2}\sum_{m_1} \frac{(-1)^{1/2-m_1}}{\sqrt{2}} \langle \tfrac{1}{2}\, n_1\, \tfrac{1}{2}\, n_2 | 1\, 1 \rangle c^{\dagger}_{m_1 n_1 m_k} c^{\dagger}_{-m_1 n_2 -m_k}.
$$

The remaining Clebsch–Gordan coefficient in this expression vanishes unless $n_1 = n_2 = \frac{1}{2}$, and $\langle \frac{1}{2}\, \frac{1}{2}\, \frac{1}{2}\, \frac{1}{2} | 1\, 1 \rangle = 1$ so

$$
A_{01}^{\dagger 01} = \sum_{m_k}\sum_{m_1} \frac{(-1)^{1/2-m_1}}{\sqrt{2}} c^{\dagger}_{m_1 \frac{1}{2} m_k} c^{\dagger}_{-m_1 \frac{1}{2} -m_k}.
$$

Writing the two terms in the sum over m_1 out explicitly for $m_1 = (-\frac{1}{2}, \frac{1}{2})$ gives

$$
\begin{aligned}
A_{01}^{\dagger 01} &= \frac{1}{\sqrt{2}}\sum_{m_k} c^{\dagger}_{\frac{1}{2}\frac{1}{2}m_k} c^{\dagger}_{-\frac{1}{2}\frac{1}{2}-m_k} - \frac{1}{\sqrt{2}}\sum_{m_k} c^{\dagger}_{-\frac{1}{2}\frac{1}{2}m_k} c^{\dagger}_{\frac{1}{2}\frac{1}{2}-m_k} \\
&= \frac{1}{\sqrt{2}}\sum_{m_k} c^{\dagger}_{1m_k} c^{\dagger}_{2-m_k} - \frac{1}{\sqrt{2}}\sum_{m_k} c^{\dagger}_{2m_k} c^{\dagger}_{1-m_k} \\
&= \frac{1}{\sqrt{2}}A^{\dagger}_{12} + \frac{1}{\sqrt{2}}A^{\dagger}_{12} \\
&= \sqrt{2}A^{\dagger}_{12}.
\end{aligned} \tag{6}
$$

3. Coupled representation for particle–hole operators

It is desirable to express the particle–hole generators of Eq. (120) in coupled representation. Let us begin by introducing a set of operators

$$P^r_\mu = \sum_{m_j m_l} (-1)^{\frac{3}{2}+m_\ell} \langle \tfrac{3}{2}\, m_j\, \tfrac{3}{2}\, m_\ell | \, r\ \mu \rangle B_{m_j - m_\ell}, \tag{7}$$

with the definition

$$B_{m_j - m_\ell} \equiv \sum_{m_k} c^\dagger_{m_j m_k} c_{-m_\ell m_k} - \frac{1}{4} \delta_{m_j - m_\ell} \Omega, \tag{8}$$

where m_j and m_ℓ take the values of the fictitious angular momentum projection m_i in the table of Fig. 25(a), providing a labeling equivalent to that of a and b in B_{ab}, with m_j or m_ℓ values $\left\{\frac{3}{2}, \frac{1}{2}, -\frac{1}{2}, -\frac{3}{2}\right\}$ mapping to a or b values $\{1,2,3,4\}$, respectively. For example, from the table of Fig. 25(a), $B_{ab} = B_{12}$ and $B_{m_j m_\ell} = B_{3/2,1/2}$ label the same quantity, which is defined in Eq. (120). From the standard selection rules for coupling of angular momentum, the index r in Eq. (7) [this document] can take the values $r = 0,1,2,3$, with $2r+1$ projections μ for each possibility, which gives a total of 16 operators P^r_μ. By inserting the explicit values of the Clebsch–Gordan coefficients the P^r_μ may be evaluated in terms of the B_{ab}. For example,

$$\begin{aligned}
P^0_0 &= \sum_{m_j m_l} (-1)^{\frac{3}{2}+m_l} \langle \tfrac{3}{2}\, m_j\, \tfrac{3}{2}\, m_l | 0\, 0 \rangle B_{m_j - m_l} \\
&= \frac{1}{2} (B_{-3/2,-3/2} + B_{-1/2,-1/2} + B_{1/2,1/2} + B_{3/2,3/2}) \\
&= \frac{1}{2} (B_{44} + B_{33} + B_{22} + B_{11}),
\end{aligned}$$

where the mapping between the labels m_i in line 2 and a in line 3 in this equation may be found in Fig. 25(a), and B_{ab} is defined in Eq. (120). Evaluating for other values of the indices gives

$$\begin{aligned}
&P^0_0 = \frac{1}{2}(B_{11}+B_{22}+B_{33}+B_{44}) = \frac{1}{2}(n_1+n_2+n_3+n_4-\Omega) = \frac{1}{2}(n-\Omega), \\
&P^1_0 = \sqrt{\frac{9}{20}}(B_{11}-B_{44}) + \sqrt{\frac{1}{20}}(B_{22}-B_{33}) = \sqrt{\frac{9}{20}}(n_1-n_4) + \sqrt{\frac{1}{20}}(n_2-n_3), \\
&P^1_1 = -\sqrt{\frac{3}{10}}B_{12} - \sqrt{\frac{4}{10}}B_{23} - \sqrt{\frac{3}{10}}B_{34} \qquad P^1_{-1} = \sqrt{\frac{3}{10}}B_{21} + \sqrt{\frac{4}{10}}B_{32} + \sqrt{\frac{3}{10}}B_{43}, \\
&P^2_0 = \frac{1}{2}(B_{11}-B_{22}+B_{44}-B_{33}) = \frac{1}{2}(n_1-n_2+n_4-n_3) \qquad P^2_1 = \frac{1}{\sqrt{2}}(B_{34}-B_{12}), \\
&P^2_{-1} = \frac{1}{\sqrt{2}}(B_{21}-B_{43}) \qquad P^2_2 = -\frac{1}{\sqrt{2}}(B_{13}+B_{24}) \qquad P^2_{-2} = -\frac{1}{\sqrt{2}}(B_{31}+B_{42}), \\
&P^3_0 = \sqrt{\frac{1}{20}}(B_{11}-B_{44}) + \sqrt{\frac{9}{20}}(B_{33}-B_{22}) = \sqrt{\frac{1}{20}}(n_1-n_4) - \sqrt{\frac{9}{20}}(n_2-n_3), \\
&P^3_1 = -\sqrt{\frac{1}{5}}B_{12} + \sqrt{\frac{3}{5}}B_{23} - \sqrt{\frac{1}{5}}B_{34} \qquad P^3_{-1} = \sqrt{\frac{1}{5}}B_{21} - \sqrt{\frac{3}{5}}B_{32} + \sqrt{\frac{1}{5}}B_{43}, \\
&P^3_2 = \sqrt{\frac{1}{2}}(B_{24}-B_{13}) \qquad P^3_{-2} = \sqrt{\frac{1}{2}}(B_{42}-B_{31}) \qquad P^3_3 = -B_{14} \qquad P^3_{-3} = B_{41}.
\end{aligned} \tag{9}$$

where the quantities n_i given by

$$n_i = B_{ii} = \sum_{m_k} c^\dagger_{i m_k} c_{i m_k} - \frac{1}{4}\Omega \tag{10}$$

are number operators for each of the four states and the total particle number n is the sum over the four states labeled by a in the table of Fig. 25(a), $n = n_1+n_2+n_3+n_4 =$ total particle number. It will be convenient to sometimes replace the operator P^0_0 with the operator S_0, according to

$$S_0 \equiv \frac{1}{2}(n-\Omega) = P^0_0, \tag{11}$$

where 2Ω is the degeneracy of the space for the particles that participate in the SO(8) symmetry. Physically $S_0 = \frac{1}{2}(n-\Omega)$ is one half the particle number measured from half filling (which corresponds to $n=\Omega$).

4. Transformations between different basis states

In exploring dynamical symmetries of the SO(8) algebra in Eq. (121) it is often useful to use different basis sets for the generators. This section gathers the relationships between the different sets of basis vectors employed in the main text. For brevity in the following, $\{P^1, P^2, P^3, S_0, S, S^\dagger, D_\mu, D_\mu^\dagger\}$ will be termed the nuclear SO(8) basis and $\{\mathcal{S}_\alpha, T_\alpha, N_\alpha, \Pi_{\alpha x}, \Pi_{\alpha y}, S_0, S, S^\dagger, D_\mu, D_\mu^\dagger\}$ will be termed the graphene SO(8) basis.

First note that the SO(8) particle–hole operators (120) can be replaced by the operators of Eq. (110) through a comparison of their definitions. For example, consider the spin operator $\mathcal{S}_y$. From Eq. (110),

$$
\begin{aligned}
\mathcal{S}_y &= \sum_{m_k}\sum_{\tau\sigma\sigma'} \langle\sigma'|\,\sigma_y\,|\sigma\rangle\, c^\dagger_{\tau\sigma' m_k} c_{\tau\sigma m_k} \\
&= \sum_{m_k}\left(-ic^\dagger_{+\uparrow m_k}c_{+\downarrow m_k} + ic^\dagger_{+\downarrow m_k}c_{+\uparrow m_k} - ic^\dagger_{-\uparrow m_k}c_{-\downarrow m_k} + ic^\dagger_{-\downarrow m_k}c_{-\uparrow m_k}\right) \\
&= -iB_{12} + iB_{21} - iB_{34} + iB_{43},
\end{aligned}
$$

where the standard 2×2 Pauli matrix representation for $\sigma_2 = \sigma_y$ was employed and equivalences between the indices a and (τ,σ) in the table of Fig. 25(a) were used to map to indices for B_{ab}. The results for the complete set of operators are displayed in Eqs. (122)-(125).

In transforming from the nuclear SO(8) basis to the graphene SO(8) basis the particle number (charge) operator n or S_0 and the 12 pairing operators $\{D_\mu, D_\mu^\dagger, S, S^\dagger\}$ are retained, but the 15 SU(4) generators $\{P^1, P^2, P^3\}$ in the nuclear representation are replaced with the 15 SU(4) generators $\{\mathcal{S}_\alpha, T_\alpha, N_\alpha, \Pi_{\alpha x}, \Pi_{\alpha y}\}$ defined in the graphene representation of Eq. (110). The explicit transformation from the $\{P^1, P^2, P^3\}$ generators to the $\{\mathcal{S}_\alpha, T_\alpha, N_\alpha, \Pi_{\alpha x}, \Pi_{\alpha y}\}$ generators is given by

$$
\begin{gathered}
\mathcal{S}_x = \sqrt{\frac{6}{5}}\left(P^1_{-1} - P^1_1\right) + \frac{2}{\sqrt{5}}\left(P^3_{-1} - P^3_1\right) \qquad \mathcal{S}_y = i\left[\sqrt{\frac{6}{5}}\left(P^1_1 + P^1_{-1}\right) + \frac{2}{\sqrt{5}}\left(P^3_1 + P^3_{-1}\right)\right], \\
\mathcal{S}_z = \frac{2}{\sqrt{5}}P^1_0 + \frac{4}{\sqrt{5}}P^3_0 = n_1 - n_2 + n_3 - n_4 \qquad T_x = -\sqrt{2}\left(P^2_2 + P^2_{-2}\right) \qquad T_y = i\sqrt{2}\left(P^2_2 - P^2_{-2}\right), \\
T_z = \frac{4}{\sqrt{5}}P^1_0 - \frac{2}{\sqrt{5}}P^3_0 = n_1 + n_2 - n_3 - n_4 \qquad N_x = \frac{1}{\sqrt{2}}\left(P^2_{-1} - P^2_1\right) \qquad N_y = \frac{i}{\sqrt{2}}\left(P^2_{-1} + P^2_1\right), \\
N_z = P^2_0 = n_1 - n_2 + n_4 - n_3 \qquad \Pi_{xx} = \frac{1}{2}\left[P^3_{-3} - P^3_3 + \sqrt{\frac{2}{5}}\left(P^1_{-1} - P^1_1\right) + \sqrt{\frac{3}{5}}\left(P^3_1 - P^3_{-1}\right)\right], \\
\Pi_{yx} = \frac{i}{2}\left[\sqrt{\frac{2}{5}}P^3_{-3} + P^3_3 + \sqrt{\frac{2}{5}}\left(P^1_{-1} + P^1_1\right) - \sqrt{\frac{3}{5}}\left(P^3_1 + P^3_{-1}\right)\right] \qquad \Pi_{zx} = -\frac{1}{\sqrt{2}}\left(P^3_2 + P^3_{-2}\right), \\
\Pi_{xy} = \frac{i}{2}\left[\sqrt{\frac{2}{5}}P^3_{-3} + P^3_3 - \sqrt{\frac{2}{5}}\left(P^1_{-1} + P^1_1\right) + \sqrt{\frac{3}{5}}\left(P^3_1 + P^3_{-1}\right)\right] \qquad \Pi_{zy} = -\frac{i}{\sqrt{2}}\left(P^3_2 - P^3_{-2}\right), \\
\Pi_{yy} = \frac{1}{2}\left[-P^3_{-3} + P^3_3 - \sqrt{\frac{2}{5}}\left(P^1_1 - P^1_{-1}\right) + \sqrt{\frac{3}{5}}\left(P^3_1 - P^3_{-1}\right)\right].
\end{gathered} \tag{12}
$$

In Eqs. (122)=(125) (and in Eqs. 12 [this document]) the graphene basis $\{\mathcal{S}_\alpha, T_\alpha, N_\alpha, \Pi_{\alpha x}, \Pi_{\alpha y}\}$ has been expressed in terms of the generators B_{ab} defined in Eq. (120). The inverse transformations giving the B_{ab} generators in terms of the $\{\mathcal{S}_\alpha, T_\alpha, N_\alpha, \Pi_{\alpha x}, \Pi_{\alpha y}\}$ generators are [23]

$$
\begin{aligned}
B_{12} &= \frac{1}{2}N_x + \frac{1}{2}iN_y + \frac{1}{4}\mathcal{S}_x + \frac{1}{4}i\mathcal{S}_y & B_{13} &= \frac{1}{4}T_x + \frac{1}{4}iT_y + \frac{1}{2}\Pi_{zx} - \frac{1}{2}i\Pi_{zy}, \\
B_{14} &= \frac{1}{2}\Pi_{xx} - \frac{1}{2}i\Pi_{yx} - \frac{1}{2}i\Pi_{xy} - \frac{1}{2}\Pi_{yy} & B_{23} &= \frac{1}{2}\Pi_{xx} + \frac{1}{2}i\Pi_{yx} - \frac{1}{2}i\Pi_{xy} + \frac{1}{2}\Pi_{yy}, \\
B_{24} &= \frac{1}{4}T_x + \frac{1}{4}iT_y - \frac{1}{2}\Pi_{zx} + \frac{1}{2}i\Pi_{zy} & B_{34} &= \frac{1}{4}\mathcal{S}_x - \frac{1}{2}iN_y - \frac{1}{2}N_x + \frac{1}{4}i\mathcal{S}_y, \\
B_{11} &= \frac{1}{4}\mathcal{S}_z + \frac{1}{4}T_z + \frac{1}{2}N_z + \frac{1}{4}(n-\Omega) & B_{22} &= -\frac{1}{4}\mathcal{S}_z + \frac{1}{4}T_z - \frac{1}{2}N_z + \frac{1}{4}(n-\Omega), \\
B_{33} &= \frac{1}{4}\mathcal{S}_z - \frac{1}{4}T_z - \frac{1}{2}N_z + \frac{1}{4}(n-\Omega) & B_{44} &= -\frac{1}{4}\mathcal{S}_z - \frac{1}{4}T_z + \frac{1}{2}N_z + \frac{1}{4}(n-\Omega),
\end{aligned} \tag{13}
$$

where the unlisted operators may be obtained from $B_{ba} = B^\dagger_{ab}$ and the diagonal operators have been assumed to obey the U(4) constraint

$$B_{11} + B_{22} + B_{33} + B_{44} = n - \Omega, \tag{14}$$

with $n = n_1 + n_2 + n_3 + n_4$ the total particle number and Ω the total pair degeneracy given by Eq. (104).

5. Lie algebra in the nuclear SO(8) basis

Because the six operators defined by Eq. (127), their six hermitian conjugates, and the 16 operators defined by Eq. (7) [this document] are independent linear combinations of the SO(8) generators defined in Eqs. (119) and (120), the 28 operators $\{P^\ell_\mu, S, S^\dagger, D_\mu, D^\dagger_\mu\}$ also close an SO(8) algebra under commutation. The SO(8) commutation relations for the coupled-representation generators

$$G'_{\mathrm{SO(8)}} = \{P^1, P^2, P^3, S_0, S, S^\dagger, D_\mu, D^\dagger_\mu\}$$

in Eq. (152) are given explicitly by [31,48]

$$[S, S^\dagger] = -2S_0, \tag{15a}$$

$$[D_{\mu'}, D^\dagger_\mu] = -2\delta_{\mu\mu'} S_0 + \sum_{t \text{ odd}} (-1)^{\mu'} \langle 2, -\mu'\, 2\, \mu | t, \mu - \mu' \rangle \begin{Bmatrix} 2 & 2 & t \\ \frac32 & \frac32 & \frac32 \end{Bmatrix} P^t_{\mu,-\mu'}, \tag{15b}$$

$$[D^\dagger_\mu, S] = P^2_\mu, \tag{15c}$$

$$[P^r_\mu, S^\dagger] = 2\delta_{r2} D^\dagger_\mu + 2\delta_{r0}\delta_{\mu 0} S^\dagger, \tag{15d}$$

$$[P^r_{\mu'}, D^\dagger_\mu] = 2(-1)^{\mu'} \delta_{r2}\delta_{-\mu\mu'} - 4\sqrt{5(2r+1)} \langle r\, \mu'\, 2\, \mu | 2, \mu + \mu' \rangle \begin{Bmatrix} 2 & 2 & r \\ \frac32 & \frac32 & \frac32 \end{Bmatrix} D^\dagger_{\mu+\mu'}, \tag{15e}$$

$$[P^r_{\mu'}, P^s_\mu] = 2(-1)^{r+s} \sqrt{(2r+1)(2s+1)} \sum_t \langle r\, \mu'\, s\, \mu | t, \mu + \mu' \rangle \left[1 - (-1)^{r+s+t}\right] \begin{Bmatrix} r & s & t \\ \frac32 & \frac32 & \frac32 \end{Bmatrix} P^t_{\mu+\mu'}, \tag{15f}$$

where S_0 is defined in Eq. (11) and $\{\,\}$ denotes the Wigner 6-j symbol [49] for the recoupling of three angular momenta to good total angular momentum.

6. Tables

For convenience we include below Table I [this document] of Clebsch–Gordan coefficients and Table II [this document] of 3J-symbols, with the Clebsch–Gordan coefficients $\langle j_1\, m_1\, j_2\, m_2 | J\, M \rangle$ and 3J-symbols related by

$$\begin{pmatrix} j_1 & j_2 & J \\ m_1 & m_2 & -M \end{pmatrix} = \frac{(-1)^{j_1 - j_2 + M}}{\sqrt{2J+1}} \langle j_1\, m_1\, j_2\, m_2 | J\, M \rangle. \tag{16}$$

The values of these vector coupling coefficients are useful in various proofs contained in this Supplement.

TABLE I: Some SO(3) Clebsch–Gordan Coefficients $\langle j_1 m_1 j_2 m_2 | JM \rangle$ from Ref. [4]

j_1	j_2	m_1	m_2	J	M	CG	j_1	j_2	m_1	m_2	J	M	CG
1/2	1/2	1/2	1/2	1	1	1	1/2	1/2	1/2	−1/2	1	0	$\sqrt{1/2}$
1/2	1/2	1/2	−1/2	0	0	$\sqrt{1/2}$	1/2	1/2	−1/2	1/2	1	0	$\sqrt{1/2}$
1/2	1/2	−1/2	1/2	0	0	$-\sqrt{1/2}$	1/2	1/2	−1/2	−1/2	1	−1	1
1	1/2	1	1/2	3/2	3/2	1	1	1/2	1	−1/2	3/2	1/2	$\sqrt{1/3}$
1	1/2	1	−1/2	1/2	1/2	$\sqrt{2/3}$	1	1/2	0	1/2	3/2	1/2	$\sqrt{2/3}$
1	1/2	0	1/2	1/2	1/2	$-\sqrt{1/3}$	1	1/2	0	−1/2	3/2	−1/2	$\sqrt{2/3}$
1	1/2	0	−1/2	1/2	−1/2	$\sqrt{1/3}$	1	1/2	−1	1/2	3/2	−1/2	$\sqrt{1/3}$
1	1/2	−1	1/2	1/2	−1/2	$-\sqrt{2/3}$	1	1/2	−1	−1/2	3/2	−3/2	1
1	1	1	1	2	2	1	1	1	1	0	2	1	$\sqrt{1/2}$
1	1	1	0	1	1	$\sqrt{1/2}$	1	1	0	1	2	1	$\sqrt{1/2}$
1	1	0	1	1	1	$-\sqrt{1/2}$	1	1	1	−1	2	0	$\sqrt{1/6}$
1	1	1	−1	1	0	$\sqrt{1/2}$	1	1	1	−1	0	0	$\sqrt{1/3}$
1	1	0	0	2	0	$\sqrt{2/3}$	1	1	0	0	1	0	0
1	1	0	0	0	0	$-\sqrt{1/3}$	1	1	−1	1	2	0	$\sqrt{1/6}$
1	1	−1	1	1	0	$-\sqrt{1/2}$	1	1	−1	1	0	0	$\sqrt{1/3}$
1	1	0	−1	2	−1	$\sqrt{1/2}$	1	1	0	−1	1	−1	$\sqrt{1/2}$
1	1	−1	0	2	−1	$\sqrt{1/2}$	1	1	−1	0	1	−1	$-\sqrt{1/2}$
1	1	−1	−1	2	−2	1							
2	1/2	2	1/2	5/2	5/2	1	2	1/2	1	−1/2	5/2	3/2	$\sqrt{1/5}$
2	1/2	2	−1/2	3/2	3/2	$\sqrt{4/5}$	2	1/2	1	1/2	5/2	3/2	$\sqrt{4/5}$
2	1/2	1	1/2	3/2	3/2	$-\sqrt{1/5}$	2	1/2	1	−1/2	5/2	1/2	$\sqrt{2/5}$
2	1/2	1	−1/2	3/2	1/2	$\sqrt{3/5}$	2	1/2	0	1/2	5/2	1/2	$\sqrt{3/5}$
2	1/2	0	1/2	3/2	1/2	$-\sqrt{2/5}$	2	1/2	0	−1/2	5/2	−1/2	$\sqrt{3/5}$
2	1/2	0	−1/2	3/2	−1/2	$\sqrt{2/5}$	2	1/2	−1	1/2	5/2	−1/2	$\sqrt{2/5}$
2	1/2	−1	1/2	3/2	−1/2	$-\sqrt{3/5}$	2	1/2	−1	−1/2	5/2	−3/2	$\sqrt{4/5}$
2	1/2	−1	−1/2	3/2	−3/2	$\sqrt{1/5}$	2	1/2	−2	1/2	5/2	−3/2	$\sqrt{1/5}$
2	1/2	−2	1/2	3/2	−3/2	$-\sqrt{4/5}$	2	1/2	−2	−1/2	5/2	−5/2	1
3/2	1/2	3/2	1/2	2	2	1	3/2	1/2	3/2	−1/2	2	1	1/2
3/2	1/2	3/2	−1/2	1	1	$\sqrt{3/4}$	3/2	1/2	1/2	1/2	2	1	$\sqrt{3/4}$
3/2	1/2	1/2	1/2	1	1	−1/2	3/2	1/2	1/2	−1/2	2	0	$\sqrt{1/2}$
3/2	1/2	1/2	−1/2	1	0	$\sqrt{1/2}$	3/2	1/2	−1/2	1/2	2	0	$\sqrt{1/2}$
3/2	1/2	−1/2	1/2	1	0	$-\sqrt{1/2}$	3/2	1/2	−1/2	−1/2	2	−1	$\sqrt{3/4}$
3/2	1/2	−1/2	−1/2	1	−1	1/2	3/2	1/2	−3/2	1/2	2	−1	1/2
3/2	1/2	−3/2	1/2	1	−1	$-\sqrt{3/4}$	3/2	1/2	−3/2	−1/2	2	−2	1

TABLE II: Some 3J coefficients $\begin{pmatrix} j_1 & j_2 & J \\ m_1 & m_2 & M \end{pmatrix}$ from Ref. [50]

$$\begin{pmatrix} a & a+1/2 & 1/2 \\ b & -b-1/2 & 1/2 \end{pmatrix} = (-1)^{a-b-1}\left[\frac{a+b+1}{(2a+1)(2a+2)}\right]^{1/2}$$

$$\begin{pmatrix} a & a & 1 \\ b & -b-1 & 1 \end{pmatrix} = (-1)^{a-b}\left[\frac{(a-b)(a+b+1)}{2a(a+1)(2a+1)}\right]^{1/2}$$

$$\begin{pmatrix} a & a & 1 \\ b & -b & 0 \end{pmatrix} = (-1)^{a-b}\frac{b}{[a(a+1)(2a+1)]^{1/2}}$$

$$\begin{pmatrix} a & a+1 & 1 \\ b & -b-1 & 1 \end{pmatrix} = (-1)^{a-b}\left[\frac{(a+b+1)(a+b+2)}{(2a+1)(2a+2)(2a+3)}\right]^{1/2}$$

$$\begin{pmatrix} a & a+1 & 1 \\ b & -b & 0 \end{pmatrix} = (-1)^{a-b-1}\left[\frac{(a-b+1)(a+b+1)}{(a+1)(2a+1)(2a+3)}\right]^{1/2}$$

$$\begin{pmatrix} a & a+1/2 & 3/2 \\ b & -b-3/2 & 3/2 \end{pmatrix} = (-1)^{a-b-1}\left[\frac{3(a+b+1)(a+b+2)(a-b)}{2a(2a+1)(2a+2)(2a+3)}\right]^{1/2}$$

$$\begin{pmatrix} a & a+1/2 & 3/2 \\ b & -b-1/2 & 1/2 \end{pmatrix} = (-1)^{a-b}(a-3b)\left[\frac{a+b+1}{2a(2a+1)(2a+2)(2a+3)}\right]^{1/2}$$

$$\begin{pmatrix} a & a+3/2 & 3/2 \\ b & -b-3/2 & 3/2 \end{pmatrix} = (-1)^{a-b-1}\left[\frac{(a+b+1)(a+b+2)(a+b+3)}{(2a+1)(2a+2)(2a+3)(2a+4)}\right]^{1/2}$$

$$\begin{pmatrix} a & a+3/2 & 3/2 \\ b & -b-1/2 & 1/2 \end{pmatrix} = (-1)^{a-b}\left[\frac{3(a-b+1)(a+b+1)(a+b+2)}{(2a+1)(2a+2)(2a+3)(2a+4)}\right]^{1/2}$$

$$\begin{pmatrix} a & a & 2 \\ b & -b-2 & 2 \end{pmatrix} = (-1)^{a-b}\left[\frac{3(a+b+1)(a+b+2)(a-b-1)(a-b)}{a(2a+3)(2a+2)(2a+1)(2a-1)}\right]^{1/2}$$

$$\begin{pmatrix} a & a & 2 \\ b & -b-1 & 1 \end{pmatrix} = (-1)^{a-b}(2b+1)\left[\frac{3(a-b)(a+b+1)}{a(2a+3)(2a+2)(2a+1)(2a-1)}\right]^{1/2}$$

$$\begin{pmatrix} a & a & 2 \\ b & -b & 0 \end{pmatrix} = (-1)^{a-b}\frac{3b^2-a(a+1)}{[a(a+1)(2a+3)(2a+1)(2a-1)]^{1/2}}$$

$$\begin{pmatrix} a & a+1 & 2 \\ b & -b-2 & 2 \end{pmatrix} = (-1)^{a-b}\left[\frac{(a+b+1)(a+b+2)(a+b+3)(a-b)}{a(a+1)(2a+4)(2a+3)(2a+1)}\right]^{1/2}$$

$$\begin{pmatrix} a & a+1 & 2 \\ b & -b-1 & 1 \end{pmatrix} = (-1)^{a-b-1}(a-2b)\left[\frac{(a+b+2)(a+b+1)}{a(a+1)(2a+4)(2a+3)(2a+1)}\right]^{1/2}$$

$$\begin{pmatrix} a & a+1 & 2 \\ b & -b & 0 \end{pmatrix} = (-1)^{a-b-1}b\left[\frac{3(a+b+1)(a-b+1)}{a(a+1)(a+2)(2a+3)(2a+1)}\right]^{1/2}$$

$$\begin{pmatrix} a & a+2 & 2 \\ b & -b-2 & 2 \end{pmatrix} = (-1)^{a-b}\left[\frac{(a+b+1)(a+b+2)(a+b+3)(a+b+4)}{(2a+1)(2a+2)(2a+3)(2a+4)(2a+5)}\right]^{1/2}$$

$$\begin{pmatrix} a & a+2 & 2 \\ b & -b-1 & 1 \end{pmatrix} = (-1)^{a-b-1}\left[\frac{(a+b+1)(a+b+2)(a+b+3)(a-b+1)}{(a+1)(a+2)(2a+1)(2a+3)(2a+5)}\right]^{1/2}$$

$$\begin{pmatrix} a & a+2 & 2 \\ b & -b & 0 \end{pmatrix} = (-1)^{a-b}\left[\frac{3(a+b+1)(a+b+2)(a-b+1)(a-b+2)}{(a+1)(2a+5)(2a+4)(2a+3)(2a+1)}\right]^{1/2}$$