Physics 541
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Dr. Guidry
Test 2 Solutions

1. Let's place the point charge on the positive z axis a distance d from the origin; then we have axial symmetry about the z axis and the solution can be expanding in Legendre polynomials rather than spherical harmonics. For the potential inside the sphere we take

$$\Phi_{\rm in} = \frac{q}{4\pi\varepsilon}\sum_l A_l \left(\frac{r}{a}\right) P_l(\cos\theta),$$

where the $q/4\pi\varepsilon$ factor is for convenience and we take a scaled radial coordinate r/a. The solution outside the sphere is taken to have the form

$$\Phi_{\rm out} = \frac{1}{4\pi\varepsilon_0}\frac{q}{|\boldsymbol{x} - d\,\hat{\boldsymbol{z}}|} + \Phi_0,$$

where the first term is the potential for the point charge [see Eq. (3.111)] and Φ_0 is a solution of the Laplace equation, $\nabla^2\Phi_0 = 0$. From Eqs. (3.97) and (3.111), we may expand $\Phi_{\rm out}$ in Legendre polynomials as

$$\Phi_{\rm out} = \frac{q}{4\pi\varepsilon_0}\sum_l \left[\frac{r_<^l}{r_>^{l+1}} + B_l\left(\frac{a}{r}\right)^{l+1}\right] P_l(\cos\theta)$$

where since $r_< = \min(r,d)$ and $r_> = \max(r,d)$ and we must match the solutions at $r = a$, we take $r_< = a$ and $r_> = d$ when using $\Phi_{\rm out}$ in the matching equations. For the electric field the matching conditions are

$$E_\theta^{\rm in} = -\frac{1}{r}\frac{\partial\Phi_{\rm in}}{\partial\theta}\bigg|_{r=a} = \frac{q}{4\pi\varepsilon}\sum_l \frac{A_l}{a} P_l'(\cos\theta)\sin\theta,$$

$$E_\theta^{\rm out} = -\frac{1}{r}\frac{\partial\Phi_{\rm out}}{\partial\theta}\bigg|_{r=a} = \frac{q}{4\pi\varepsilon_0}\sum_l \left[\frac{a^{l-1}}{d^{l+1}} + \frac{B_l}{a}\right] P_l'(\cos\theta)\sin\theta,$$

where $P_l'(\cos\theta) \equiv dP_l/d\theta$. Matching these gives the constraint

$$A_l = \frac{\varepsilon}{\varepsilon_0}\left(\frac{a^l}{d^{l+1}} + B_l\right).$$

For the displacement the matching conditions are

$$D_r^{\rm in} = -\varepsilon\frac{\partial\Phi_{\rm in}}{\partial r}\bigg|_{r=a} = \frac{q}{4\pi}\sum_l \frac{lA_l}{a} P_l(\cos\theta),$$

$$D_r^{\rm out} = -\varepsilon_0\frac{\partial\Phi_{\rm out}}{\partial r}\bigg|_{r=a} = \frac{q}{4\pi}\sum_l \left(\frac{la^{l-1}}{d^{l+1}} - \frac{l(l+1)B_l}{a}\right) P_l(\cos\theta),$$

and matching these gives the constraint

$$A_l = \frac{a^l}{d^{l+1}} - \frac{l+1}{l}B_l.$$

Thus solving the two constraint equations

$$A_l = \frac{\varepsilon}{\varepsilon_0}\left(\frac{a^l}{d^{l+1}} + B_l\right) \qquad A_l = \frac{a^l}{d^{l+1}} - \frac{l+1}{l}B_l$$

simultaneously gives

$$A_l = \frac{2l+1}{l+\frac{\varepsilon_0}{\varepsilon}(l+1)}\frac{a^l}{d^{l+1}},$$
$$B_l = \frac{(\frac{\varepsilon_0}{\varepsilon}-1)l}{l+\frac{\varepsilon_0}{\varepsilon}(l+1)}\frac{a^l}{d^{l+1}}.$$

Substituting, the potentials are then

$$\Phi_{\text{in}} = \frac{q}{4\pi\varepsilon}\sum_l \frac{2l+1}{l+\frac{\varepsilon_0}{\varepsilon}(l+1)}\frac{r^l}{d^{l+1}}P_l(\cos\theta),$$
$$\Phi_{\text{out}} = \frac{q}{4\pi\varepsilon_0}\sum_l \left[\frac{r_<^l}{r_>^{l+1}} + \frac{(\frac{\varepsilon_0}{\varepsilon}-1)l}{l+\frac{\varepsilon_0}{\varepsilon}(l+1)}\frac{a^{2l+1}}{(rd)^{l+1}}\right]P_l(\cos\theta).$$

2. (a) The battery is disconnected, so assume that the voltage V and charge on the plates Q are constant. From Eq. (3.8), the energy to charge the capacitor without the dielectric layer is

$$U_0 = \frac{1}{2}C_0V^2 = \frac{A\varepsilon_0}{2d}V^2$$

and from Eq. (4.31) the energy to charge the capacitor with the dielectric layer in place is

$$U_{\text{f}} = \frac{1}{2}C_{\text{f}}V^2 = \frac{A\varepsilon}{2d}V^2.$$

Thus, the work done on the dielectric is $W = U_{\text{f}} - U_0$, which is positive, since $\varepsilon > \varepsilon_0$. Thus the dielectric is drawn into the space between the plates.

(b) Now the battery is connected, so V may be assumed constant but the charge Q on the capacitor places can be increased by the battery. Again, the energy to charge the capacitor at constant Q without the dielectric layer is

$$U_0 = \frac{A\varepsilon_0}{2d}V^2$$

and the energy to charge the capacitor at constant Q with the dielectric layer in place is

$$U_{\text{f}} = \frac{A\varepsilon}{2d}V^2.$$

But now charge can be transferred from the battery to the plates. Since $C = Q/V$, the charge with no dielectric is

$$Q_0 = C_0V = \frac{\varepsilon_0 A}{d}V$$

and with dielectric,

$$Q_{\text{f}} = C_{\text{f}}V = \frac{\varepsilon A}{d}V.$$

Thus the work W_Q done in transferring additional charge to the plates is

$$W_Q = \frac{1}{2}(\varepsilon - \varepsilon_0)\frac{AV^2}{d}.$$

and the total work is

$$\begin{aligned} W &= (U_{\rm f} - U_0) + W_Q \\ &= \frac{AV^2}{2d}\,\varepsilon - \frac{AV^2}{2d}\,\varepsilon_0 + \frac{AV^2}{2d}\,\varepsilon - \frac{AV^2}{2d}\varepsilon_0 \\ &= \frac{AV^2}{2d}(2\varepsilon - 2\varepsilon_0) \\ &= \frac{AV^2}{d}(\varepsilon - \varepsilon_0). \end{aligned}$$

This is positive, since $\varepsilon > \varepsilon_0$, so the dielectric is pulled into space between the plates in this case also.

3. For current density $\boldsymbol{J}$ In Coulomb gauge the vector potential (5.29) is

$$\boldsymbol{A}(\boldsymbol{x}) = \frac{\mu_0}{4\pi}\int \frac{\boldsymbol{J}(\boldsymbol{x}')}{|\boldsymbol{x}-\boldsymbol{x}'|}\,d^3x'.$$

The current distribution is in the azimuthal direction by hypothesis,

$$\boldsymbol{J}(r',\theta',\phi') = J_\phi(r',\theta')\hat{\boldsymbol{\phi}}'$$

so $\boldsymbol{A}$ will have only an azimuthal component $A_\phi(r,\theta)$. Choosing an observation point with $\phi = 0$,

$$A_\phi(r,\theta) = \frac{\mu_0}{4\pi}\int \frac{J_\phi(r',\theta')\,\hat{\boldsymbol{\phi}}'\cdot\hat{\boldsymbol{\phi}}}{|\boldsymbol{x}-\boldsymbol{x}'|}\,d^3x' = \frac{\mu_0}{4\pi}\int \frac{J_\phi(r',\theta')\cos\phi'}{|\boldsymbol{x}-\boldsymbol{x}'|}\,d^3x'.$$

Expanding the denominator in spherical harmonics using Eq. (3.114),

$$A_\phi(r,\theta) = \frac{\mu_0}{4\pi}\sum_{lm}\frac{4\pi}{2l+1}Y_{lm}(\theta,0)\int \frac{r_<^l}{r_>^{l+1}}\,J_\phi(r',\theta')Y_{lm}^*(\theta',\phi')\cos\phi'\,d^3x'.$$

Replacing spherical harmonic inside the integral with an associated Legendre polynomial using Eq. (3.117), this can be written

$$\begin{aligned} A_\phi(r,\theta) = {}& \frac{\mu_0}{4\pi}\sum_{lm}\frac{4\pi}{2l+1}Y_{lm}(\theta,0) \\ & \times(-1)^m\sqrt{\frac{2l+1}{4\pi}\frac{(l-m)!}{(l+m)!}}\int \frac{r_<^l}{r_>^{l+1}}\,J_\phi(r',\theta')P_l^m(\cos\theta')e^{-im\phi'}\cos\phi'\,d^3x'. \end{aligned}$$

Evaluating the integral over $d\phi'$ restricts m to ± 1,

$$\int_0^{2\pi} e^{-im\phi'}\cos\theta'd\phi' = \pi\left(\delta_{m,1}+\delta_{m,-1}\right)$$

and using Eqs. (3.117) and (3.120), the $m = \pm 1$ terms are equal for each l. Thus, converting the spherical harmonic $Y_{lm}(\theta, 0)$ outside the integral also to an associated Legendre polynomial,

$$A_\phi(r,\theta) = \frac{\mu_0}{4\pi}\sum_l \frac{1}{l(l+1)} P_l^1(\cos\theta) \int \frac{r_<^l}{r_>^{l+1}} P_l^1(\cos\theta') J_\phi(r',\theta')\, d^3x'.$$

For the interior solution $(r < r')$ this becomes,

$$A_\phi^{\text{in}}(r,\theta) = \frac{\mu_0}{4\pi}\sum_l \frac{r^l}{l(l+1)} P_l^1(\cos\theta) \int (r')^{-l-1} P_l^1(\cos\theta') J_\phi(r',\theta')\, d^3x',$$

while it becomes

$$A_\phi^{\text{out}}(r,\theta) = \frac{\mu_0}{4\pi}\sum_l \frac{1}{l(l+1)r^{l+1}} P_l^1(\cos\theta) \int (r')^{l} P_l^1(\cos\theta') J_\phi(r',\theta')\, d^3x',$$

for the exterior solution $(r > r')$.

4. The Coulomb gauge condition is that $\boldsymbol{\nabla}\cdot\boldsymbol{A} = 0$. If

$$\boldsymbol{A}(\boldsymbol{x}) = \frac{\mu_0}{4\pi}\int \frac{\boldsymbol{J}(\boldsymbol{x}')}{|\boldsymbol{x}-\boldsymbol{x}'|}\, d^3x',$$

then the divergence is

$$\begin{aligned}
\boldsymbol{\nabla}\cdot\boldsymbol{A} &= \frac{\mu_0}{4\pi}\int \boldsymbol{J}(\boldsymbol{x}')\cdot\boldsymbol{\nabla}_x \frac{1}{|\boldsymbol{x}-\boldsymbol{x}'|}\, d^3x' \\
&= -\frac{\mu_0}{4\pi}\int \boldsymbol{J}(\boldsymbol{x}')\cdot\boldsymbol{\nabla}_{x'} \frac{1}{|\boldsymbol{x}-\boldsymbol{x}'|}\, d^3x' \\
&= \frac{\mu_0}{4\pi}\int \frac{\boldsymbol{\nabla}_{x'}\cdot\boldsymbol{J}(\boldsymbol{x}')}{|\boldsymbol{x}-\boldsymbol{x}'|}\, d^3x' \\
&= 0,
\end{aligned}$$

since charge-current conservation requires that $\boldsymbol{\nabla}\cdot\boldsymbol{J} = 0$, from Eq. (1.3).