Physics 541
Spring, 2024
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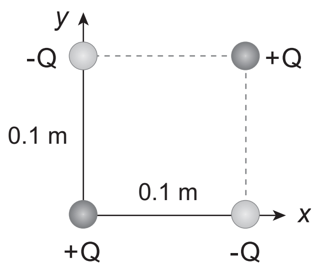
Test 1 Solutions

1. For the charge distribution



the (traceless) quadrupole moment has cartesian components

$$Q_{ij} \equiv \int (3x'_i x'_j - r'^2 \delta_{ij})\, \rho(\boldsymbol{x}') d^3x' = \sum_{k=1}^{N} q_k (3x_i x_j - r^2 \delta_{ij})$$

where $r'^2 \equiv |\boldsymbol{x}'|^2$ and we've used $\rho(\boldsymbol{x}) = \sum_{i=1}^{N} q_i \delta(\boldsymbol{x}' - \boldsymbol{x}_i)$. Assuming the origin to be at the lower left charge, that $|Q| = 3\,\mu$C, and that $d = 0.1$ m, the quadrupole moment cartesian components are

$$\begin{aligned}
Q_{xx} &= \sum_{i=1}^{N} q_i (3x_i x_i - r_i^2) \\
&= Q\left(3[0^2 - d^2 + d^2 - 0^2] - [0 - d^2 + 2d^2 - d^2]\right) = 0, \\
Q_{xy} &= \sum_{i=1}^{N} q_i (3x_i y_i) = Q \times 3d^2 = 3 \times 10^{-6}\ \mathrm{C} = 9 \times 10^{-8}\ \mathrm{C{\cdot}m^2} \\
Q_{yx} &= \sum_{i=1}^{N} q_i (3y_i x_i) = Q \times 3d^2 = 9 \times 10^{-8}\ \mathrm{C{\cdot}m^2} \\
Q_{yy} &= \sum_{i=1}^{N} q_i (3y_i y_i - r_i^2) = Q\left([3d^2 - 3d^2] - [-d^2 + 2d^2 - d^2]\right) = 0, \\
Q_{zz} &= Q_{xz} = Q_{zx} = Q_{yz} = Q_{zy} = 0 \qquad \text{(since } z = 0 \text{ for all charges),}
\end{aligned}$$

where we have used that $r^2 = x^2 + y^2$. Note that $\mathrm{Tr}\, Q = Q_{xx} + Q_{yy} + Q_{zz} = 0$ and $Q_{ij} = Q_{ji}$.

2. For separation of variables in the axially-symmetric Laplace equation the most general solution is given by Eq. (3.96),

$$\Phi(r, \theta) = \sum_{l=0}^{\infty} A_l r^l P_l(\cos\theta) + \sum_{l=0}^{\infty} B_l r^{-(l+1)} P_l(\cos\theta).$$

For the inside solution ($r \leq R$) the B_l terms vary as $1/r^{l+1}$ and are unbounded at the origin and must be discarded, so

$$\Phi(r,\theta) = \sum_{l=0}^{\infty} A_l r^l P_l(\cos\theta) \qquad (r \leq R).$$

For the outside solution ($r > R$) the A_l terms vary as r^l and are unbounded for large r and must be discarded, leaving

$$\Phi(r,\theta) = \sum_{l=0}^{\infty} B_l r^{-(l+1)} P_l(\cos\theta) \qquad (r > R).$$

Next we require the solutions to be continuous at $r = R$, implying that

$$\sum_{l=0}^{\infty} A_l R^l P_l(\cos\theta) = \sum_{l=0}^{\infty} B_l R^{-(l+1)} P_l(\cos\theta),$$

which is satisfied if

$$B_l = R^{2l+1} A_l.$$

From Eq. (3.11), the radial derivative of Φ has a discontinuity when a charge layer is crossed,

$$\boldsymbol{\nabla}\Phi_{\text{above}} - \boldsymbol{\nabla}\Phi_{\text{below}} = -\frac{\sigma}{\varepsilon_0}\hat{\boldsymbol{n}},$$

which means that the derivatives of the inside and outside solutions must satisfy

$$\left(\frac{\partial\Phi_{\text{outside}}}{\partial r} - \frac{\partial\Phi_{\text{inside}}}{\partial r}\right)\bigg|_{r=R} = -\frac{\sigma(\theta)}{\varepsilon_0}.$$

Evaluating the derivatives,

$$-\sum_{l=0}^{\infty}(l+1)\frac{B_l}{R^{l+2}}P_l(\cos\theta) - \sum_{l=0}^{\infty} l A_l R^{l-1} P_l(\cos\theta) = -\frac{\sigma(\theta)}{\varepsilon_0},$$

and upon combining the terms on the left side using $B_l = R^{2l+1}A_l$,

$$\sum_{l=0}^{\infty}(2l+1)R^{l-1}A_i P_l(\cos\theta) = \frac{\sigma(\theta)}{\varepsilon_0}.$$

Finally, we can determine the coefficients A_l and $B_l = R^{2l+1}A_l$ using the Legendre polynomial orthogonality condition: multiply the preceding equation by $P_k(\cos\theta)\sin\theta d\theta$ and integrate from 0 to π to give

$$\sum_{l=0}^{\infty}(2l+1)R^{l-1}A_l \int_0^{\pi} P_l(\cos\theta)P_k(\cos\theta)\sin\theta d\theta = \frac{1}{\varepsilon_0}\int_0^{\pi}\sigma(\theta)P_k(\cos\theta)\sin\theta d\theta.$$

By virtue of the Legendre polynomial orthogonality condition of Eq. (3.98),

$$\int_0^{\pi} P_l(\cos\theta)P_k(\cos\theta)\sin\theta d\theta = \begin{cases} \dfrac{2}{2l+1} & (\text{if } l = k), \\ 0 & (\text{if } l \neq k), \end{cases}$$

the only term that survives in the sum on the left side is for $l = k$, giving

$$(2l+1)R^{l-1}A_l \frac{2}{2l+1} = \frac{1}{\varepsilon_0}\int_0^{\pi} \sigma(\theta)P_k(\cos\theta)\sin\theta d\theta,$$

so the coefficients A_l are determined by

$$A_l = \frac{1}{2\varepsilon_0 R^{l-1}}\int_0^{\pi} \sigma(\theta)P_l(\cos\theta)\sin\theta d\theta,$$

from which the coefficients B_l may be obtained using $B_l = R^{2l+1}A_l$. Thus the interior and exterior solutions

$$\Phi(r,\theta) = \sum_{l=0}^{\infty} A_l r^l P_l(\cos\theta) \qquad (r \le R).$$

$$\Phi(r,\theta) = \sum_{l=0}^{\infty} B_l r^{-(l+1)} P_l(\cos\theta) \qquad (r > R).$$

are now given in terms of infinite sums with known coefficients A_l and B_l for all terms in the sums.

3. The potential is

$$\Phi(\boldsymbol{x}) = \frac{1}{4\pi\varepsilon_0}\int \rho(\boldsymbol{x}')\frac{1}{|\boldsymbol{x}-\boldsymbol{x}'|}d^3x'.$$

Inserting the effective density

$$\rho_{\text{eff}}(\boldsymbol{x}) = -\boldsymbol{p}\cdot\boldsymbol{\nabla}\delta(\boldsymbol{x}-\boldsymbol{x}_0),$$

for $\rho(\boldsymbol{x}')$, the potential is

$$\begin{aligned}
\Phi(\boldsymbol{x}) &= -\frac{1}{4\pi\varepsilon_0}\int \boldsymbol{p}\cdot\boldsymbol{\nabla}_{x'}\delta(\boldsymbol{x}'-\boldsymbol{x}_0)\frac{1}{|\boldsymbol{x}-\boldsymbol{x}'|}d^3x' \\
&= \frac{1}{4\pi\varepsilon_0}\int \delta(\boldsymbol{x}'-\boldsymbol{x}_0)\boldsymbol{p}\cdot\boldsymbol{\nabla}_{x'}\frac{1}{|\boldsymbol{x}-\boldsymbol{x}'|}d^3x' \\
&= \frac{1}{4\pi\varepsilon_0}\boldsymbol{p}\cdot\boldsymbol{\nabla}_{x'}\frac{1}{|\boldsymbol{x}-\boldsymbol{x}'|}\bigg|_{x'=x_0} \\
&= \frac{1}{4\pi\varepsilon_0}\frac{\boldsymbol{p}\cdot(\boldsymbol{x}-\boldsymbol{x}_0)}{|\boldsymbol{x}=\boldsymbol{x}_0)|^3},
\end{aligned}$$

where $\boldsymbol{\nabla}_{x'}$ means the gradient with respect to $\boldsymbol{x}'$, in line 2 we have used that the delta function anticommutes with derivatives [see Eq. (A.61)], in line 3 we've invoked the basic property of the delta function, and in line 4 we have used that

$$\boldsymbol{\nabla}_{x'}\left(\frac{1}{|\boldsymbol{x}-\boldsymbol{x}'|}\right) = \frac{\boldsymbol{x}-\boldsymbol{x}'}{|\boldsymbol{x}-\boldsymbol{x}'|^3},$$

from Eq. (A.11b). This is the expected potential for a dipole at $\boldsymbol{x}_0$. Likewise, Eq. (3.123) gives for the energy of this dipole in an external field,

$$W = \int \rho(\boldsymbol{x})\Phi(\boldsymbol{x})d^3x.$$

Inserting $\rho_{\text{eff}}(\boldsymbol{x})$ for $\rho(\boldsymbol{x}')$,

$$\begin{aligned} W &= -\int \boldsymbol{p}\cdot\boldsymbol{\nabla}\delta(\boldsymbol{x}-\boldsymbol{x}_0)\,\Phi(\boldsymbol{x})\,d^3x \\ &= \int \delta(\boldsymbol{x}-\boldsymbol{x}_0)\boldsymbol{p}\cdot\boldsymbol{\nabla}\,\Phi(\boldsymbol{x})\,d^3x \\ &= \boldsymbol{p}\cdot\boldsymbol{\nabla}\Phi(\boldsymbol{x})\big|_{\boldsymbol{x}=\boldsymbol{x}_0} \\ &= \boldsymbol{p}\cdot\boldsymbol{\nabla}\Phi(\boldsymbol{x}_0) \\ &= -\boldsymbol{p}\cdot\boldsymbol{E}(\boldsymbol{x}_0), \end{aligned}$$

where in line 2 the delta function has been anticommuted with the derivative operator $\boldsymbol{\nabla}$, in line 3 the basic property of the delta function has been invoked, and in the last line we have used $E = -\boldsymbol{\nabla}\Phi$. This is the dipole interaction energy with an external field appearing in the multipole expansion of Eq. (3.125).

4. From Eq. (3.119), the quadrupole moment tensor is

$$Q_{ij} \equiv \int (3x'_i x'_j - r'^2\delta_{ij})\,\rho(\boldsymbol{x}')d^3x',$$

where $r'^2 \equiv |\boldsymbol{x}'|^2$. This formula for a continuous charge distribution $\rho(\boldsymbol{x}')$ can be converted into a formula for N discrete point charges q_i using

$$\rho(\boldsymbol{x}) = \sum_{i=1}^{N} q_i\delta(\boldsymbol{x}'-\boldsymbol{x}_i),$$

so that

$$\begin{aligned} Q_{ij} &= \int \sum_{k=1}^{N} q_k\delta(\boldsymbol{x}'-\boldsymbol{x}_k)(3x'_i x'_j - r'^2\delta_{ij})\,d^3\boldsymbol{x}' \\ &= \sum_{k=1}^{N} q_k(3x_i x_j - r^2\delta_{ij}). \end{aligned}$$

Thus letting i be the index for the discrete charges and $k = \{1,2,3\} = \{x,y,z\}$ be the cartesian component index, the trace (sum over diagonal elements) is

$$\begin{aligned} \operatorname{Tr} Q &= \sum_{i=1}^{3} Q_{ii} \\ &= \sum_{i=1}^{3}\sum_{k=1}^{N} q_k(3x_{ik}x_{ik} - r_k^2) \\ &= \sum_{k=1}^{N} q_k\sum_{i=1}^{3}(3x_{ik}x_{ik} - r_k^2) \\ &= \sum_{k=1}^{N} q_k(3r_k^2 - 3r_k^2) = 0, \end{aligned}$$

where $\sum_{i=1}^{3} r_k^2 = x^2+y^2+z^2$ has been used.