5 Magnetostatics in Vacuum

5.1 For current density $\boldsymbol{J}$ In Coulomb gauge the vector potential (5.29) is

$$\boldsymbol{A}(\boldsymbol{x}) = \frac{\mu_0}{4\pi}\int \frac{\boldsymbol{J}(\boldsymbol{x}')}{|\boldsymbol{x}-\boldsymbol{x}'|}\, d^3x'.$$

The current distribution is in the azimuthal direction by hypothesis,

$$\boldsymbol{J}(r',\theta',\phi') = J_\phi(r',\theta')\hat{\boldsymbol{\phi}}'$$

so $\boldsymbol{A}$ will have only an azimuthal component $A_\phi(r,\theta)$. Choosing an observation point with $\phi = 0$,

$$A_\phi(r,\theta) = \frac{\mu_0}{4\pi}\int \frac{J_\phi(r',\theta')\,\hat{\boldsymbol{\phi}}'\cdot\hat{\boldsymbol{\phi}}}{|\boldsymbol{x}-\boldsymbol{x}'|}\, d^3x' = \frac{\mu_0}{4\pi}\int \frac{J_\phi(r',\theta')\cos\phi'}{|\boldsymbol{x}-\boldsymbol{x}'|}\, d^3x'.$$

Expanding the denominator in spherical harmonics using Eq. (3.114),

$$A_\phi(r,\theta) = \frac{\mu_0}{4\pi}\sum_{lm}\frac{4\pi}{2l+1}Y_{lm}(\theta,0)\int \frac{r_<^l}{r_>^{l+1}}J_\phi(r',\theta')Y_{lm}^*(\theta',\phi')\cos\phi'\, d^3x'.$$

Replacing spherical harmonic inside the integral with an associated Legendre polynomial using Eq. (3.117), this can be written

$$A_\phi(r,\theta) = \frac{\mu_0}{4\pi}\sum_{lm}\frac{4\pi}{2l+1}Y_{lm}(\theta,0)$$
$$\times(-1)^m\sqrt{\frac{2l+1}{4\pi}\frac{(l-m)!}{(l+m)!}}\int \frac{r_<^l}{r_>^{l+1}}J_\phi(r',\theta')P_l^m(\cos\theta')e^{-im\phi'}\cos\phi'\, d^3x'.$$

Evaluating the integral over $d\phi'$ restricts m to ± 1,

$$\int_0^{2\pi} e^{-im\phi'}\cos\theta' d\phi' = \pi\left(\delta_{m,1}+\delta_{m,-1}\right)$$

and using Eqs. (3.117) and (3.120), the $m = \pm 1$ terms are equal for each l. Thus, converting the spherical harmonic $Y_{lm}(\theta,0)$ outside the integral also to an associated Legendre polynomial,

$$A_\phi(r,\theta) = \frac{\mu_0}{4\pi}\sum_l \frac{1}{l(l+1)}P_l^1(\cos\theta)\int \frac{r_<^l}{r_>^{l+1}}P_l^1(\cos\theta')J_\phi(r',\theta')\, d^3x'.$$

For the interior solution $(r < r')$ this becomes,

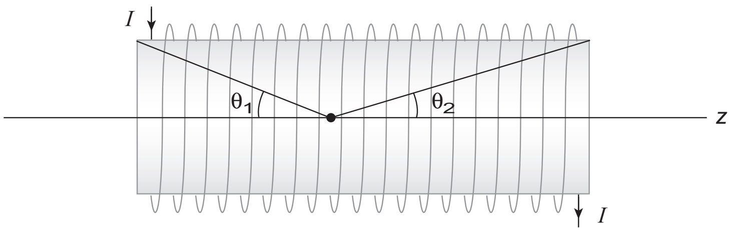
$$A_\phi^{\text{in}}(r,\theta) = \frac{\mu_0}{4\pi}\sum_l \frac{r^l}{l(l+1)}P_l^1(\cos\theta)\int (r')^{-l-1}P_l^1(\cos\theta')J_\phi(r',\theta')\, d^3x',$$

while it becomes

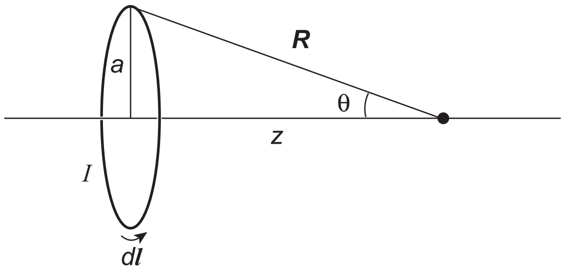
$$A_\phi^{\text{out}}(r,\theta) = \frac{\mu_0}{4\pi}\sum_l \frac{1}{l(l+1)r^{l+1}} P_l^1(\cos\theta) \int (r')^l P_l^1(\cos\theta') J_\phi(r',\theta')\, d^3x',$$

for the exterior solution $(r > r')$.

5.2 The solenoid



of length L and radius a carries a current I through N turns per unit length. First consider a single loop of radius a and current I,



Applying the Biot–Savart law (5.7) to the single loop,

$$B = \frac{\mu_0 I}{4\pi}\int \frac{d\boldsymbol{l}\times\boldsymbol{R}}{|\boldsymbol{R}|^3}.$$

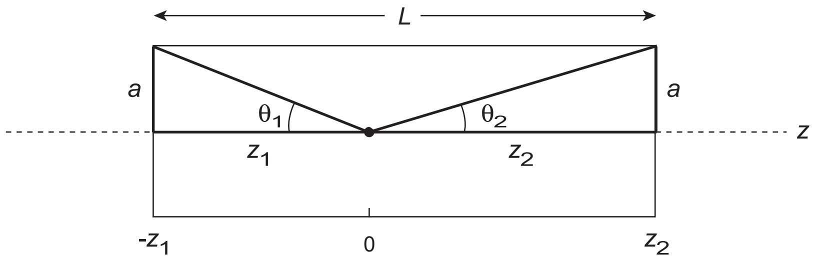
But by symmetry the magnetic field will be along the horizontal (z) axis, and

$$d\boldsymbol{l}\times\boldsymbol{R} = dl\,R\sin\theta = dl\,R\left(\frac{a}{R}\right) = a\,dl,$$

where $R = \sqrt{a^2+z^2}$. Then,

$$\begin{aligned} B_z &= \frac{\mu_0 I}{4\pi}\int \frac{a}{R^3}\,dl \\ &= \frac{\mu_0 I}{4\pi}\frac{a}{R^3}(2\pi a) \\ &= \frac{\mu_0 I}{2}\frac{a^2}{(a^2+z^2)^{3/2}}. \end{aligned}$$

Now use linear superposition of many rings, taking $NL \to \infty$ so that we can assume each ring is perpendicular to the axis. Then from the following diagram,



where $z_1 + z_2 = L$, for N coils

$$B_z = \frac{\mu_0 I a^2}{2} \int_{-z_1}^{z_2} \frac{N\,dz}{(a^2+z^2)^{3/2}}.$$

To perform the integral, make the trig substitution,

$$z = a\tan\theta \qquad dz = \frac{a}{\cos^2\theta}\,d\theta,$$

which gives

$$\begin{aligned} B_z &= \frac{\mu_0 NI}{2} \int_{-\tan^{-1}(z_1/a)}^{\tan^{-1}(z_2/a)} \cos\theta\,d\theta \\ &= \frac{\mu_0 NI}{2} \sin\theta \Big|_{-\tan^{-1}(z_1/a)}^{\tan^{-1}(z_2/a)} \\ &= \frac{\mu_0 NI}{2} \left(\frac{z_2}{a^2+z_2^2} - \frac{z_1}{a^2+z_1^2} \right) \\ &= \frac{\mu_0 NI}{2} (\cos\theta_1 + \cos\theta_2), \end{aligned}$$

where we have used

$$\sin(\tan^{-1}x) = \frac{x}{\sqrt{1+x^2}} \qquad \cos\theta_1 = -\frac{z_1}{a^2+z_1^2} \qquad \cos\theta_2 = \frac{z_2}{a^2+z_2^2}$$

(see geometry of diagram above).