# 4 Electrostatics in Matter

**4.2** From the solution of Problem 3.5, the inside solution $(r \leq R)$ is

$$\Phi(r,\theta) = \sum_{l=0}^{\infty} A_l r^l P_l(\cos\theta) \qquad (r \leq R),$$

and the outside solution $(r > R)$ is

$$\Phi(r,\theta) = \sum_{l=0}^{\infty} B_l r^{-(l+1)} P_l(\cos\theta) \qquad (r \geq R),$$

where the coefficients $A_l$ are determined by

$$A_l = \frac{1}{2\varepsilon_0 R^{l-1}} \int_0^{\pi} \sigma(\theta) P_l(\cos\theta) \sin\theta d\theta,$$

from which the coefficients $B_l$ may be obtained using $B_l = R^{2l+1} A_l$.

If $\sigma(\theta) = kP_1(\cos\theta) = k\cos\theta$ for a constant $k$, then all $A_l$ values vanish except for

$$A_1 = \frac{k}{2\varepsilon_0} \int_0^{\pi} [P_1(\cos\theta)]^2 \sin\theta d\theta = \frac{k}{3\varepsilon_0},$$

from which the only non-vanishing $B_i$ is

$$B_1 = R^{2(1)+1} A_1 = \frac{kR^3}{3\varepsilon_0}.$$

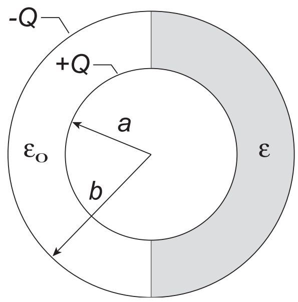
Thus inside the sphere

$$\Phi(r,\theta) = A_1 r P_1(\cos\theta) = \frac{kr}{3\varepsilon_0}\cos\theta \qquad (r \leq R),$$

and outside the sphere

$$\Phi(r,\theta) = B_1 r^{-2} P_l(\cos\theta) = \frac{kR^3}{3\varepsilon_0 r^2}\cos\theta \qquad (r \geq R).$$

**4.3** The concentric spheres with dielectric in gray:




(a) Obviously without the dielectric the electric field between the spheres would be radial,

$$\boldsymbol{E} = \frac{Q}{4\pi\varepsilon_0}\frac{\hat{\boldsymbol{r}}}{r^2},$$

but we would expect the dielectric to modify this. However, notice that a radial field is tangential to the boundary between the dielectric and open region so the field matching conditions for discontinuities given in Eqs. (4.53) and (4.54) are automatically satisfied, suggesting that a solution with radial symmetry may still be valid in the presence of the dielectric. Thus we try a solution of the form

$$\boldsymbol{E} = C\frac{\hat{\boldsymbol{r}}}{r^2}.$$

The constant $C$ can be determined using Gauss's law in medium (4.47a):

$$\oint_S \boldsymbol{D}\cdot\boldsymbol{n}\,da = \int_V \rho\, d^3x = Q.$$

Using $\boldsymbol{D} = \varepsilon_0\boldsymbol{E}$ from Eq. (4.25) and that $\boldsymbol{E}$ is directed radially,

$$\begin{aligned}
\oint_S \boldsymbol{D}\cdot\boldsymbol{n}\,da &= \frac{1}{2}\varepsilon_0\oint_S \boldsymbol{E}\cdot\boldsymbol{n}\,da + \frac{1}{2}\varepsilon\oint_S \boldsymbol{E}\cdot\boldsymbol{n}\,da \\
&= \frac{1}{2}\varepsilon_0|\boldsymbol{E}|\oint_S da + \frac{1}{2}\varepsilon|\boldsymbol{E}|\oint_S da \\
&= \frac{\varepsilon_0 C}{2r^2}(4\pi r^2) + \frac{\varepsilon C}{2r^2}(4\pi r^2) \\
&= 2\pi(\varepsilon_0+\varepsilon)C \\
&= Q.
\end{aligned}$$

Therefore, $C = Q/2\pi(\varepsilon_0+\varepsilon)$ and

$$\boldsymbol{E} = \frac{Q}{2\pi(\varepsilon_0+\varepsilon)}\frac{\hat{\boldsymbol{r}}}{r^2}.$$

(b) The boundary matching condition is given by Eq. (4.53a),

$$(\boldsymbol{D}_2 - \boldsymbol{D}_1)\cdot\boldsymbol{n} = \sigma,$$

where $\boldsymbol{n}$ is a unit normal to the boundary surface and $\sigma$ is the surface charge. But the inner sphere is conducting so there is no electric field inside and $\boldsymbol{D}_1 = \varepsilon_0 \boldsymbol{E} = 0$. Thus, letting $\boldsymbol{D}^{\perp} \equiv \boldsymbol{D}_1 \cdot \boldsymbol{n}$,

$$\sigma = \boldsymbol{D}^{\perp}\Big|_{r=a}$$

Therefore, on the half of the inner sphere without dielectric outside it $\boldsymbol{D}^{\perp} = \varepsilon_0 \boldsymbol{E}^{\perp}$ and

$$\sigma = \varepsilon_0 \boldsymbol{E}^{\perp}\Big|_{r=a} = \varepsilon_0 \frac{Q}{2\pi(\varepsilon_0+\varepsilon)r^2}\Big|_{r=a} = \frac{\varepsilon_0}{2\pi(\varepsilon_0+\varepsilon)}\frac{Q}{a^2} \qquad \text{(half with no dielectric).}$$

while on the half of the inner sphere with dielectric outside it,

$$\sigma = \varepsilon \boldsymbol{E}^{\perp}\Big|_{r=a} = \varepsilon \frac{Q}{2\pi(\varepsilon_0+\varepsilon)r^2}\Big|_{r=a} = \frac{\varepsilon}{2\pi(\varepsilon_0+\varepsilon)}\frac{Q}{a^2} \qquad \text{(half with dielectric).}$$

(c) The surface polarization-charge density is give by Eq. (4.40),

$$\sigma_{\text{pol}} = \boldsymbol{P} \cdot \boldsymbol{n} \equiv -P^{\perp},$$

where the polarization $\boldsymbol{P}$ is given by Eq. (4.27),

$$\boldsymbol{P} = (\varepsilon - \varepsilon_0)\boldsymbol{E}.$$

In the hemisphere with the dielectric the polarization surface charge is

$$\sigma_{\text{pol}} = -P^{\perp}\Big|_{r=a} = -\,(\varepsilon-\varepsilon_0)E^{\perp}\Big|_{r=a} = -\frac{\varepsilon-\varepsilon_0}{2\pi(\varepsilon+\varepsilon_0)}\frac{Q}{a^2},$$

while in the hemisphere without the dielectric $\sigma_{\text{pol}} = 0$. Notice that in the hemisphere without the dielectric the total charge density (free + polarization) is

$$\begin{aligned}\sigma_{\text{total}} &= \sigma + \sigma_{\text{pol}} \\ &= \frac{\varepsilon_0}{2\pi(\varepsilon_0+\varepsilon)}\frac{Q}{a^2} + 0 \\ &= \frac{\varepsilon_0}{2\pi(\varepsilon_0+\varepsilon)}\frac{Q}{a^2} \qquad \text{(half with no dielectric),}\end{aligned}$$

while in the hemisphere with the dielectric

$$\begin{aligned}\sigma_{\text{total}} &= \sigma + \sigma_{\text{pol}} \\ &= \frac{\varepsilon}{2\pi(\varepsilon_0+\varepsilon)}\frac{Q}{a^2} - \frac{\varepsilon-\varepsilon_0}{2\pi(\varepsilon+\varepsilon_0)}\frac{Q}{a^2} \\ &= \frac{\varepsilon_0}{2\pi(\varepsilon_0+\varepsilon)}\frac{Q}{a^2} \qquad \text{(half with dielectric).}\end{aligned}$$

Thus the total surface charge density is the same on either half of the sphere, which is why the electric field is radially symmetric.

**4.4** (a) Assume the inner conductor is positively charged with a charge of $\lambda$ per unit length. The capacitance is $C = Q/V$ where $V$ is the potential difference and Q the total charge. By Gauss's law applied to the inner cylinder the magnitude of the electric field is $E = \lambda/2\pi r\varepsilon_0$

and by symmetry it is normal to the surface, so the potential difference between the inner and outer cylinder is

$$V = \int_a^b E dr = \int_a^b \frac{\lambda}{2\pi\varepsilon_0 r} dr = \frac{\lambda}{2\pi\varepsilon_0} \ln\left(\frac{b}{a}\right).$$

Therefore, the capacitance per unit length $C'$ is

$$C' = \frac{\lambda}{V} = \frac{\lambda}{(\lambda/2\pi\varepsilon_0)\ln(b/a)} = \frac{2\pi\varepsilon_0}{\ln(b/a)}.$$

(b) At equilibrium the electrostatic force pulling the dielectric liquid up between the cylinders is just balanced by the gravitational force acting down on the liquid. The electrostatic force is

$$F = \frac{dW}{ds} = \frac{1}{2}V^2\frac{dC}{ds},$$

where $V$ is the total potential difference between the cylinders, we've used $W = \frac{1}{2}CV^2$ from Eq. (3.7), and $C$ is the total capacitance

$$C = C_h + C_{L-h},$$

where $C_h$ is the capacitance for the section with dielectric liquid between the electrodes and $C_{L-h}$ is the capacitance of the section above that with air between the electrodes. From Eq. (4.2), a capacitance $C_0$ is modified by dielectric material between its electrodes according to

$$C = \kappa C_0 = (1+\chi_\mathrm{e})C_0$$

where $\kappa$ is the dielectric constant and $\chi_\mathrm{e}$ is the electric susceptibility of the dielectric material, and as shown in part (a), the capacitance per unit length is

$$C' = \frac{2\pi\varepsilon_0}{\ln(b/a)}$$

for a cylindrical capacitor in vacuum. Then, neglecting the susceptibility of air,

$$\begin{aligned} C &= C_h + C_{L-h} \\ &= \frac{2\pi\varepsilon_0(1+\chi_\mathrm{e})h}{\ln(b/a)} + \frac{2\pi\varepsilon_0(L-h)}{\ln(b/a)} \\ &= \frac{2\pi\varepsilon_\mathrm{e}}{\ln(b/a)}(\chi_\mathrm{e}h + L). \end{aligned}$$

From this the electrostatic force is

$$F_\mathrm{e} = \frac{1}{2}V^2\frac{dC}{dh} = \frac{\pi\varepsilon_0\chi_\mathrm{e}V^2}{\ln(b/a)},$$

and the gravitational force may be computed as

$$F_\mathrm{g} = mg = \pi\rho(b_2 - a^2)hg,$$

where $\rho$ is the mass density and $m$ is the mass of the dielectric liquid between the electrodes. Therefore, setting $F_e = F_g$ and solving for $\chi_e$ gives

$$\chi_e = \frac{hg\rho(b^2 - a^2)\ln(b/a)}{\varepsilon_0 V^2},$$

for the electric susceptibility of the dielectric liquid.

# 5 Magnetostatics in Vacuum

**5.1** For current density $\boldsymbol{J}$ In Coulomb gauge the vector potential (5.29) is

$$\boldsymbol{A}(\boldsymbol{x}) = \frac{\mu_0}{4\pi}\int \frac{\boldsymbol{J}(\boldsymbol{x}')}{|\boldsymbol{x}-\boldsymbol{x}'|}\,d^3x'.$$

The current distribution is in the azimuthal direction by hypothesis,

$$\boldsymbol{J}(r',\theta',\phi') = J_\phi(r',\theta')\hat{\boldsymbol{\phi}}'$$

so $\boldsymbol{A}$ will have only an azimuthal component $A_\phi(r,\theta)$. Choosing an observation point with $\phi = 0$,

$$A_\phi(r,\theta) = \frac{\mu_0}{4\pi}\int \frac{J_\phi(r',\theta')\,\hat{\boldsymbol{\phi}}'\cdot\hat{\boldsymbol{\phi}}}{|\boldsymbol{x}-\boldsymbol{x}'|}\,d^3x' = \frac{\mu_0}{4\pi}\int \frac{J_\phi(r',\theta')\cos\phi'}{|\boldsymbol{x}-\boldsymbol{x}'|}\,d^3x'.$$

Expanding the denominator in spherical harmonics using Eq. (3.114),

$$A_\phi(r,\theta) = \frac{\mu_0}{4\pi}\sum_{lm}\frac{4\pi}{2l+1}Y_{lm}(\theta,0)\int \frac{r_<^l}{r_>^{l+1}}J_\phi(r',\theta')Y_{lm}^*(\theta',\phi')\cos\phi'\,d^3x'.$$

Replacing spherical harmonic inside the integral with an associated Legendre polynomial using Eq. (3.117), this can be written

$$A_\phi(r,\theta) = \frac{\mu_0}{4\pi}\sum_{lm}\frac{4\pi}{2l+1}Y_{lm}(\theta,0)$$
$$\times(-1)^m\sqrt{\frac{2l+1}{4\pi}\frac{(l-m)!}{(l+m)!}}\int \frac{r_<^l}{r_>^{l+1}}J_\phi(r',\theta')P_l^m(\cos\theta')e^{-im\phi'}\cos\phi'\,d^3x'.$$

Evaluating the integral over $d\phi'$ restricts $m$ to $\pm 1$,

$$\int_0^{2\pi} e^{-im\phi'}\cos\theta'\,d\phi' = \pi\,(\delta_{m,1}+\delta_{m,-1})$$

and using Eqs. (3.117) and (3.120), the $m = \pm 1$ terms are equal for each $l$. Thus, converting the spherical harmonic $Y_{lm}(\theta,0)$ outside the integral also to an associated Legendre polynomial,

$$A_\phi(r,\theta) = \frac{\mu_0}{4\pi}\sum_l \frac{1}{l(l+1)}P_l^1(\cos\theta)\int \frac{r_<^l}{r_>^{l+1}}P_l^1(\cos\theta')J_\phi(r',\theta')\,d^3x'.$$

For the interior solution $(r < r')$ this becomes,

$$A_\phi^{\text{in}}(r,\theta) = \frac{\mu_0}{4\pi}\sum_l \frac{r^l}{l(l+1)}P_l^1(\cos\theta)\int (r')^{-l-1}P_l^1(\cos\theta')J_\phi(r',\theta')\,d^3x',$$