3 Solutions of Poisson and Laplace Equations

3.1 See Fig. 3.1 [this document]. In the analog problem of Fig. 3.1(b) a second point charge q' has been added at a distance $b = R^2/a$ to the right of the center of the sphere in (a), with $q' = -(R/a)q$ (where the location b and the charge q' have been chosen so that the potential is $\Phi = 0$ on the sphere in the analog problem). The potential for the analog problem in spherical coordinates (r, θ) is

$$\begin{aligned}\Phi(\boldsymbol{r}) &= \frac{1}{4\pi\varepsilon_0}\left(\frac{q}{s} + \frac{q'}{r'}\right) \\ &= \frac{1}{4\pi\varepsilon_0}\left(\frac{q}{(r^2 + a^2 - 2ra\cos\theta)^{1/2}} + \frac{q}{(R^2 + (ra/R)^2 - 2ra\cos\theta)^{1/2}}\right),\end{aligned}$$

where the law of cosines (A.50) and $q' = -(R/a)q$ have been used in the second line. But on the sphere $r = R$ and $\Phi = 0$. Thus, the analog problem (b) has the same boundary conditions as the actual problem (a) in the exterior region and the solution to the actual problem is

$$\Phi(\boldsymbol{r}) = \frac{1}{4\pi\varepsilon_0}\left(\frac{q}{s} + \frac{q'}{r'}\right).$$

The distance between q and q' is $a - b$ and the force is

$$F = \frac{1}{4\pi\varepsilon_0}\frac{qq'}{(a-b)^2} = -\frac{1}{4\pi\varepsilon_0}\frac{q^2Ra}{(a^2 - R^2)^2}.$$

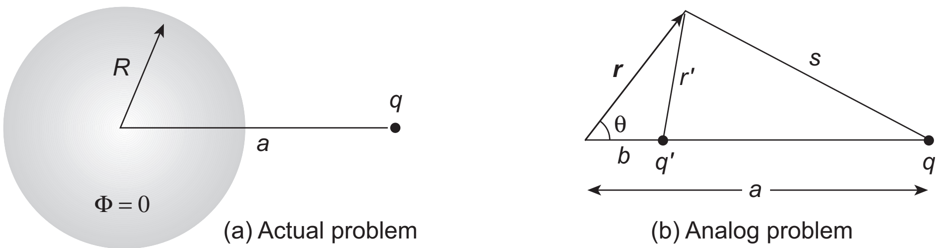
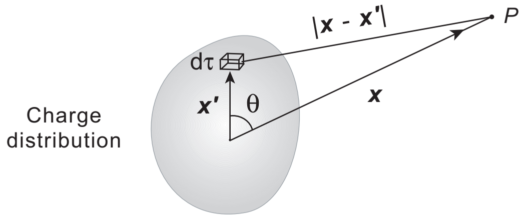


Fig. 3.1 Figure associated with Problem 3.1. (a) The actual problem of a single charge q outside a conducting sphere at ground potential. (b) Analog problem using an image charge. The lower left vertex of the triangle is at the center of the sphere in (a). The image charge q' is placed a distance $b = R^2/a$ to the right of the center of the sphere of the actual problem with $q' = -(R/a)q$.

3.2 From the following figure,



by the law of cosines (A.50),

$$R^2 = r^2 + (r')^2 - 2rr'\cos\theta = r^2\left[1 + \left(\frac{r'}{r}\right)^2 - 2\left(\frac{r'}{r}\right)\cos\theta,\right]$$

where we define

$$r \equiv |\boldsymbol{x}| \qquad r' \equiv |\boldsymbol{x}'| \qquad R \equiv |\boldsymbol{x} - \boldsymbol{x}'|.$$

Therefore,

$$R = r(1+\varepsilon)^{1/2} \qquad \varepsilon \equiv \frac{r'}{r}\left(\frac{r'}{r} - 2\cos\theta\right).$$

Therefore,

$$\frac{1}{|\boldsymbol{x}-\boldsymbol{x}'|} = \frac{1}{R} = \frac{1}{r}(1+\varepsilon)^{-1/2} \simeq \frac{1}{r}\left((1 - \frac{1}{2}\varepsilon + \frac{3}{8}\varepsilon^2 - \frac{5}{16}\varepsilon^3 + \cdots\right)$$

where we have used that ε is small since $r \gg r'$ to justify a binomial expansion in the last step. Expanding this using the explicit definition for ε from above,

$$\frac{1}{|\boldsymbol{x}-\boldsymbol{x}'|} = \frac{1}{r}\left[1 + \frac{r'}{r}\cos\theta + \left(\frac{r'}{r}\right)^2\left(\frac{3\cos^2\theta - 1}{2}\right) + \left(\frac{r'}{r}\right)^3\left(\frac{5\cos^3\theta - 3\cos\theta}{2}\right) + \cdots\right].$$

But from Table 3.1 the first four Legendre polynomials are,

$$P_0 = 1 \qquad P_1 = \cos\theta \qquad P_2 = \frac{1}{2}(3\cos^2\theta - 1) \qquad P_3 = \frac{1}{2}(5\cos^3\theta - 3\cos\theta),$$

so the preceding equation is an expansion in Legendre polynomials

$$\frac{1}{|\boldsymbol{x}-\boldsymbol{x}'|} = \frac{1}{r}\sum_{l=0}^{\infty}\left(\frac{r'}{r}\right)^l P_l(\cos\theta).$$

Alternatively, using the spherical harmonic addition theorem (3.92),

$$P_l(\cos\gamma) = \frac{4\pi}{2l+1}\sum_{m=-l}^{l} Y_{lm}^*(\theta',\phi')Y_{lm}(\theta,\phi),$$

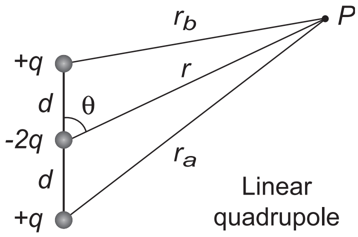
we can write

$$\frac{1}{|\boldsymbol{x}-\boldsymbol{x}'|} = 4\pi\sum_{l=0}^{\infty}\sum_{m=-l}^{l}\frac{1}{2l+1}\frac{(r')^l}{r^{l+1}}Y_{lm}^*(\theta',\phi')Y_{lm}(\theta,\phi),$$

which is Eq. (3.94) with $r_< = r'$ and $r_> = r$. From Eq. (3.90), the potential can then be written

$$\Phi(\boldsymbol{x}) = \frac{1}{4\pi\varepsilon_0}\int \frac{\rho(\boldsymbol{x}')}{|\boldsymbol{x}-\boldsymbol{x}'|}d^3x' = \frac{1}{4\pi\varepsilon_0}\sum_{l=0}^{\infty} r^{-(l+1)}\int (r')^l P_l(\cos\theta)\rho(\boldsymbol{x}')d^3x'.$$

3.3 The linear quadrupole charge distribution is.



The potential at point P is given by

$$\Phi = \frac{1}{4\pi\varepsilon_0}\left(\frac{Q}{r_a} - \frac{2Q}{r} + \frac{Q}{r_b}\right) = \frac{Q}{4\pi\varepsilon_0 r}\left(\frac{r}{r_a} + \frac{r}{r_b} - 2\right).$$

Utilizing the law of cosines to evaluate distances as for the dipole in Box 3.2, and neglecting terms of order higher than d^2/r^2 in the expansion, we find

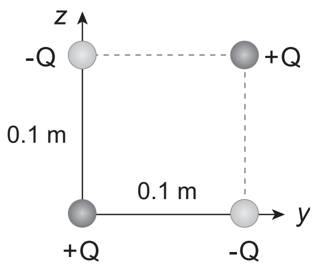
$$\frac{r}{r_a} = 1 - \frac{d}{r}\cos\theta + \frac{d^2}{r^2}\frac{(3\cos^2\theta - 1)}{2}$$
$$\frac{r}{r_b} = 1 + \frac{d}{r}\cos\theta + \frac{d^2}{r^2}\frac{(3\cos^2\theta - 1)}{2}$$

Therefore the electrostatic potential of the linear quadrupole is given by

$$\Phi = \frac{2Qd^2}{4\pi\varepsilon_0 r^3}\frac{3\cos^2\theta - 1}{2} = \left(\frac{2Qd^2}{4\pi\varepsilon_0}\right)\frac{P_2(\cos\theta)}{r^3} \qquad (r^2 \gg d^2),$$

which is proportional to $P_2(\cos\theta)$ and varies inversely as the cube of the distance.

3.4 For the charge distribution,



The dipole moment vector is

$$\boldsymbol{p} = \sum_{i=1}^{4} Q_i \boldsymbol{x}_i,$$

where i labels the charges and the magnitude of the charges is $|Q| = 3\,\mu\text{C}$. The cartesian components of the dipole moment vector are then

$$\begin{aligned}
p_x &= \sum_{i=1}^{4} Q_i x_i \\
&= (+3\,\mu\text{C})(0.0\,\text{m}) + (-3\,\mu\text{C})(0.1\,\text{m}) + (3\,\mu\text{C})(0.1\,\text{m}) + (-3\,\mu\text{C})(0.0\,\text{m}) = 0.0\,\text{C}\,\text{m}, \\
p_y &= \sum_{i=1}^{4} Q_i y_i \\
&= (+3\,\mu\text{C})(0.0\,\text{m}) + (-3\,\mu\text{C})(0.0\,\text{m}) + (3\,\mu\text{C})(0.1\,\text{m}) + (-3\,\mu\text{C})(0.1\,\text{m}) = 0.0\,\text{C}\,\text{m}, \\
p_z &= 0.0\,\text{C}\,\text{m}.
\end{aligned}$$

So this configuration has no dipole moment.