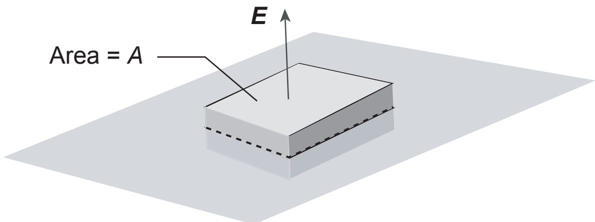
2 Electrostatics in Vacuum

2.1 Consider the rectangular-box gaussian surface in the following figure,



which lies partially above and partially below the infinite plane of charge. Applying Gauss's law to the box,

$$\oint \boldsymbol{E} \cdot d\boldsymbol{a} = \frac{1}{\varepsilon_0} Q.$$

The enclosed charge is $Q = \sigma A$, where σ is the surface charge density and A is the area of the top of the box. By symmetry $\boldsymbol{E}$ points upward for points above the plane and downward for points below the plane. Thus the top and bottom surfaces of the box contribute

$$2 \int \boldsymbol{E} \cdot d\boldsymbol{a} = 2\,|\boldsymbol{E}| \int da = 2\,|\boldsymbol{E}|\,A,$$

and the sides don't contribute since $\boldsymbol{E}$ is parallel to the sides. Thus,

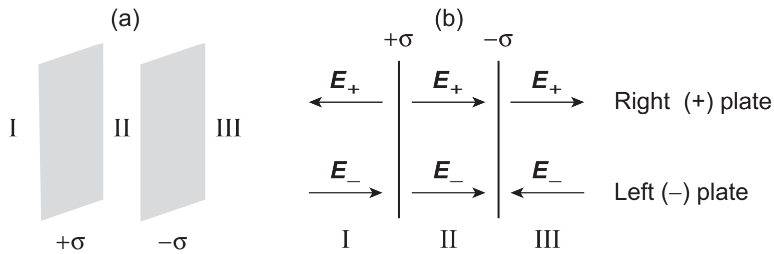
$$2A\,|\boldsymbol{E}| = \frac{1}{\varepsilon_0} \sigma A,$$

Implying that

$$\boldsymbol{E} = \frac{\sigma}{2\varepsilon_0} \hat{\boldsymbol{n}},$$

where $\hat{\boldsymbol{n}}$ is a unit vector pointing away from the surface.

2.2 The parallel infinite planes with regions I, II, and III are shown in figure (a) below:

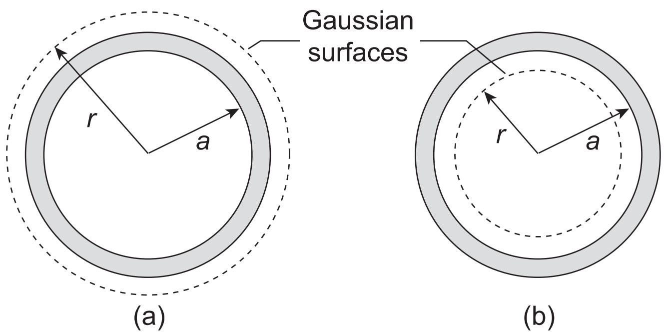


The field generated by a single infinite plane was worked out in Problem 2.1,

$$\boldsymbol{E} = \frac{\sigma}{2\varepsilon_0}\hat{\boldsymbol{n}},$$

where $\hat{\boldsymbol{n}}$ is a unit vector pointing away from the surface. The left (positively charged) plate produces a field $\sigma/2\varepsilon_0$ pointing to the left in region I and to the right in regions II and III. The right (negatively charged) plate produces a field $\sigma/2\varepsilon_0$ pointing toward the right plate, which is to the left in region III and to the right in regions I and II. Thus the two fields cancel in regions I and III and reinforce each other in region II [see figure (b) above]. So the total field is zero in regions I and III, and has magnitude $2(\sigma/2\varepsilon_0) = \sigma/\varepsilon_0$ and points to the right in region II between the plates.

2.4 For the field outside the spherical shell, consider the gaussian surface with radius $r > a$ in Fig. (a) in the following.



Then from Gauss's law,

$$\oint_S \boldsymbol{E}\cdot\boldsymbol{n}\,da = \oint_S E_n\,da = E_n\oint_S da = 4\pi r^2 E_n = \frac{Q}{\varepsilon_0},$$

where $\boldsymbol{n}$ is a unit vector normal to the surface, in the second step we have used that by symmetry the $\boldsymbol{E}$ field must point radially outward parallel to $\boldsymbol{n}$ with component E_n. Thus, *outside the shell* the field is radial with component

$$E_n = E = \frac{1}{4\pi\varepsilon_0}\frac{Q}{r^2} \qquad \text{(outside the spherical shell).}$$

Inside the shell, use the gaussian surface in Fig. (b) above with radius $r < a$. Then applying Gauss's law again,

$$\oint_S \boldsymbol{E}\cdot\boldsymbol{n}\,da = \oint_S E_n\,da = E_n\oint_S da = 4\pi r^2 E_n = \frac{Q}{\varepsilon_0} = 0,$$

since there is no charge inside the shell. Thus *inside the shell* the field is given by

$$E_n = E = \frac{1}{4\pi\varepsilon_0}\frac{Q}{r^2} = 0 \qquad \text{(inside the spherical shell),}$$

since the gaussian surface in (b) encloses no charge.

2.5 The curl of the electric field vanishes, $\boldsymbol{\nabla} \times \boldsymbol{E} = 0$. Then from Eq. (A.12)

$$\boldsymbol{\nabla} \times \boldsymbol{E} = \left(\frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z}\right)\hat{\boldsymbol{x}} + \left(\frac{\partial E_x}{\partial z} - \frac{\partial E_z}{\partial x}\right)\hat{\boldsymbol{y}} + \left(\frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y}\right)\hat{\boldsymbol{z}} = 0.$$

Since all cartesian components of $\boldsymbol{\nabla} \times \boldsymbol{E}$ must vanish for the curl to be zero, the quantities in parentheses must be equal to zero, which gives Eq. (2.31).