Chapter 21

Gravitational Waves

Just as

- Maxwell's theory permits the existence of propagating waves in the electromagnetic field,
- General relativity predicts that fluctuations in the metric of spacetime can propagate as *gravitational waves*.

Gravitational waves are of considerable current interest for three basic reasons:

- The existence of gravitational waves is the last fundamental prediction of general relativity that has not been confirmed directly.
- Gravitational waves are capable of probing the Universe back to the Planck Scale.
- Present technology may now be capable of detecting gravitational waves.

The Last Untested Prediction of General Relativity

Gravitational waves represent the last prediction of the general theory of relativity that has not been tested directly.

- Although the Binary Pulsar provides strong *indirect evidence* for the emission of gravitational waves from that system, no gravitational wave has yet been detected directly.
- This is not because gravitational waves are expected to be rare;
- it is because they are expected to interact so weakly with matter.
- General relativity is tested by the mere existence of gravitational waves and by their detailed properties.
- For example, general relativity predicts that gravitational waves can have only two states of polarization and must travel at light velocity.

Some alternative theories of gravity predict gravitational waves with as many as six polarization states and with speeds that can be less than c.

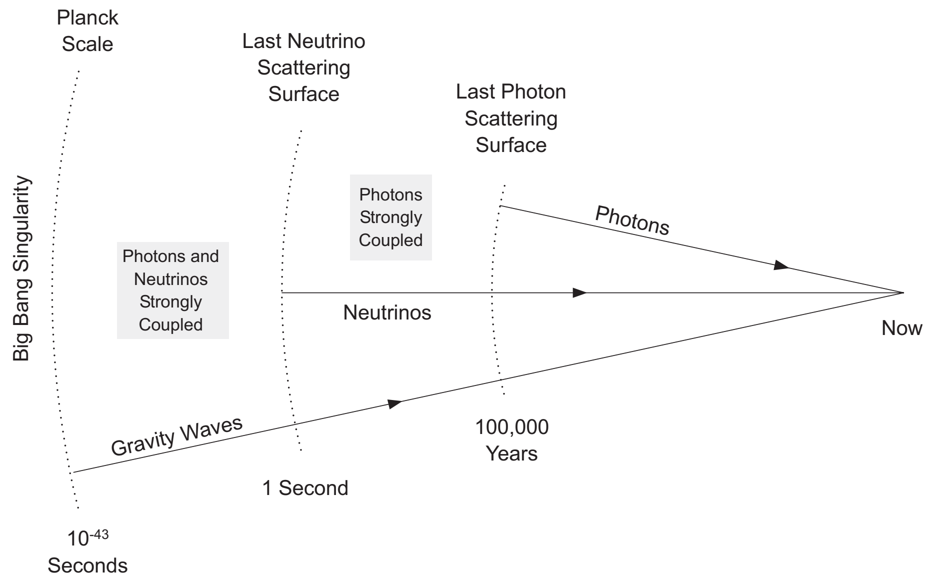


Figure 21.1: Probes of the early Universe. More weakly interacting particles can be seen from earlier epochs. In principle, gravity waves from times close to the Planck scale might be visible in the current Universe.

The Deepest Probe

Weakness of gravitational waves makes them difficult to detect but

- That same weakness means that gravitational waves can in principle be seen from earlier epoches than can be probed directly by observing other forms of radiation (Fig. 21.1).
- Photons decouple $\sim 10^5$ y after the big bang, so we can't see direct photons from earlier periods.
- Neutrinos couple more weakly and fell out of equilibrium at $\sim$ 1 s after the big bang, so neutrinos could give direct information only back to a time only $\sim$ 1 s after the big bang.
- In principle, gravitational waves could go back to the inflationary or even Planck scales.

Technology May Now Exist to Detect Gravitational Waves

> A current and proposed generation of gravity wave detectors may have sufficient sensitivity to make the first direct measurements of gravitational waves within the coming decade. Thus, there is a real chance that the currently primarily theoretical topic of gravity waves will soon become an experimental science.

21.1 Linearized Gravity

The Einstein equation may be expressed in the form

$$R_{\mu\nu} = -8\pi G(T_{\mu\nu} - \tfrac{1}{2}g_{\mu\nu}T^{\lambda}_{\lambda}).$$

- Because this equation is non-linear, general solutions that correspond to gravitational waves are difficult to obtain.
- In many instances we may assume that the gravitational waves are weak, which allows the metric to be expressed in the form

  $$g_{\mu\nu}(x) = \eta_{\mu\nu} + h_{\mu\nu}(x),$$

  where $\eta_{\mu\nu}$ is the metric of (flat) Minkowski space and $h_{\mu\nu}$ is small.
- The *linearized vacuum Einstein equation* then results from
  - inserting the approximate metric into the vacuum Einstein equation,

    $$R_{\mu\nu} = 0$$

    obtained by setting $T_{\mu\nu} = 0$, and
  - expansion of the resulting equations to first order in $h_{\mu\nu}$.

21.1.1 Linearized Curvature Tensor

The Ricci curvature tensor $R_{\mu\nu}$ is given by

$$R_{\mu\nu} = \Gamma^{\lambda}_{\mu\nu,\lambda} - \Gamma^{\lambda}_{\mu\lambda,\nu} + \Gamma^{\lambda}_{\mu\nu}\Gamma^{\sigma}_{\lambda\sigma} - \Gamma^{\sigma}_{\mu\lambda}\Gamma^{\lambda}_{\nu\sigma},$$

where the Christoffel symbols $\Gamma^{\sigma}_{\lambda\mu}$ are related to the metric tensor $g_{\mu\nu}$ by

$$\Gamma^{\sigma}_{\lambda\mu} = \tfrac{1}{2} g^{\nu\sigma} \left(\frac{\partial g_{\mu\nu}}{\partial x^{\lambda}} + \frac{\partial g_{\lambda\nu}}{\partial x^{\mu}} - \frac{\partial g_{\mu\lambda}}{\partial x^{\nu}} \right)$$

To zeroth order in $h_{\mu\nu}$, the Christoffel coefficients vanish and so does $R_{\mu\nu}$, since $\partial g / \partial x = 0$ for $\eta_{\mu\nu}$. To first order in $h_{\mu\nu}$,

$$\delta\Gamma^{\gamma}_{\mu\nu} = \frac{1}{2} \eta^{\gamma\delta} \left(\frac{\partial h_{\delta\mu}}{\partial x^{\nu}} + \frac{\partial h_{\delta\nu}}{\partial x^{\mu}} - \frac{\partial h_{\mu\nu}}{\partial x^{\delta}} \right).$$

The last two terms in the above equation for $R_{\mu\nu}$ are quadratic in ∂h and may be discarded to first order in h, giving

$$\delta R_{\mu\nu} = \frac{\partial(\delta\Gamma^{\gamma}_{\mu\nu})}{\partial x^{\gamma}} - \frac{\partial(\delta\Gamma^{\gamma}_{\mu\gamma})}{\partial x^{\nu}} + \mathcal{O}\left(h^2\right).$$

21.1.2 Wave Equation

Substitution and some algebra yields (Exercise)

$$\delta R_{\mu\nu} = \tfrac{1}{2}(-\Box h_{\mu\nu} + \partial_\mu V_\nu + \partial_\nu V_\mu),$$

where the 4-dimensional Laplacian (d'Alembertian operator) is defined by

$$\Box \equiv \eta^{\alpha\beta}\partial_\alpha\partial_\beta = -\frac{\partial^2}{\partial t^2} + \nabla^2$$

with the definitions

$$\partial_\mu \equiv \partial/\partial x^\mu \qquad \partial^\mu \equiv \partial/\partial x_\mu$$

and where the V_ν are defined through

$$\begin{aligned} V_\nu &\equiv \partial_\gamma h^\gamma_\nu - \tfrac{1}{2}\partial_\nu h^\gamma_\gamma \\ &= \partial_\gamma \eta^{\gamma\delta} h_{\delta\nu} - \tfrac{1}{2}\partial_\nu \eta^{\gamma\delta} h_{\delta\gamma}. \end{aligned}$$

Note that raising and lowering of indices in linearized gravity is generally accomplished through contraction with the flat-space metric $\eta_{\mu\nu}$,

$$h^\gamma_\nu \equiv \eta^{\gamma\delta} h_{\delta\nu}$$

rather than through contraction with $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$.

Thus the vacuum Einstein equation to this order yields the wave equation

$$\Box h_{\mu\nu} - \partial_\mu V_\nu - \partial_\nu V_\mu = 0.$$

- Since $h_{\mu\nu}$ is symmetric, this constitutes a set of 10 *linear* partial differential equations for the metric perturbation $h_{\mu\nu}$.
- One refers to these as the *linearized vacuum Einstein equations* and to the resulting theory as *linearized gravity*.

As is clear from the derivation, we may expect this to be a valid approximation to the full gravitational theory when the metric departs only slightly from that of flat spacetime.

21.1.3 Degrees of Freedom and Gauge Transformations

The equation

$$\Box h_{\mu\nu} - \partial_\mu V_\nu - \partial_\nu V_\mu = 0.$$

cannot yield unique solutions in its present form because of the freedom of coordinate transformations:

- Given one solution (metric), we may generate another by a coordinate transformation.

- This ambiguity is related to a similar ambiguity in electromagnetism that is associated with *freedom to make gauge transformations without altering the fields.*

- The symmetric tensor $h_{\mu\nu}$ has *10 independent components* but they are related by *four Bianchi identities:*

$$G^{\mu}_{\ \nu;\mu} = 0.$$

- This leaves $10 - 4 = 6$ independent equations in the 10 unknowns $h_{\mu\nu}$, implying *4 degrees of freedom.*

- These are the coordinate transformations: if $h_{\mu\nu}$ solves the linearized Einstein equations, so does $h'_{\mu\nu}$ where $h'_{\mu\nu}$ is related to $h_{\mu\nu}$ through a general coordinate transformation $x \to x'$.

- Such a transformation involves four arbitrary functions $x'^{\mu}(x)$.

This is *analogous to the gauge ambiguity in electromagnetism,* which is removed by *fixing a gauge.* Here we can remove the ambiguity by *fixing the coordinate system.* Four coordinate conditions added to the six independent equations then permit unique solutions to

$$\Box h_{\mu\nu} - \partial_\mu V_\nu - \partial_\nu V_\mu = 0.$$

Let us examine this in a little more detail.

21.1.4 Choice of Gauge

In electromagnetism it is possible to make different choices of the (vector and scalar) potentials that give the same electric and magnetic fields and thus the same classical observables.

- This freedom is associated with gauge invariance.
- Something similar is possible in linearized gravity.
- Small changes can be made in the coordinates that leave $\eta_{\mu\nu}$ unchanged in
$$g_{\mu\nu}(x) = \eta_{\mu\nu} + h_{\mu\nu}(x),$$
but that alter the functional form of $h_{\mu\nu}$.
- Under small changes in the coordinates
$$x^\mu \to x'^\mu = x^\mu + \varepsilon^\mu(x),$$
where $\varepsilon^\mu(x)$ is similar in size to $h_{\mu\nu}$.
- Then,
$$\begin{aligned} g_{\mu\nu}(x) = \eta_{\mu\nu} + h_{\nu\mu}(x) \to \eta_{\mu\nu} + h'_{\mu\nu}(x) \\ = \eta_{\mu\nu} + h_{\mu\nu}(x) - \partial_\mu \varepsilon_\nu - \partial_\nu \varepsilon_\mu . \end{aligned}$$
- The transformation $h_{\mu\nu} \to h'_{\mu\nu}$, with
$$h'_{\mu\nu} = h_{\mu\nu}(x) - \partial_\mu \varepsilon_\nu - \partial_\nu \varepsilon_\mu ,$$
is termed a *gauge transformation,* by analogy with electromagnetism. If $h_{\mu\nu}$ is a solution of
$$\Box h_{\mu\nu} - \partial_\mu V_\nu - \partial_\nu V_\mu = 0.$$
so is $h'_{\mu\nu}$.

Table 21.1: Gauge invariance in linearized gravity and electromagnetism

	Linearized Gravity	Electromagnetism†
Potentials	$h_{\mu\nu}$	Vector potential: $\boldsymbol{A}(t,\boldsymbol{x})$ Scalar potential : $\Phi(t,\boldsymbol{x})$
Fields	Linearized Riemann curvature: $R_{\alpha\beta\delta\gamma}(x)$	Electric field: $\boldsymbol{E}(t,\boldsymbol{x})$ Magnetic field: $\boldsymbol{B}(t,\boldsymbol{x})$
Gauge transformation	$h_{\mu\nu} \to h_{\mu\nu} - \partial_\mu \varepsilon_\nu - \partial_\nu \varepsilon_\mu$	$A^\mu \to A^\mu - \partial^\mu \chi$
Example of a gauge condition ("Lorentz")	$\partial_\nu h^\nu_\mu - \frac{1}{2}\partial_\mu h^\nu_\nu = 0$	$\partial^\mu A_\mu = 0$
Field equations in Lorentz gauge	$\Box h_{\mu\nu} = 0$	$\Box A^\mu = 0$

†The 4-vector potential is $A^\mu \equiv (\Phi, \boldsymbol{A})$ and χ is an arbitrary scalar function.

Just as

- gauge transformations in the Maxwell theory lead to
  - new potentials but
  - the same electric and magnetic fields,
- gauge transformations in gravity lead to
  - new potentials $h_{\mu\nu}$ but to
  - the same fields (the linearized form $\delta R_{\mu\nu\beta\gamma}(x)$ of the Riemann curvature tensor).

The analogies between coordinate ("gauge") transformations in linearized gravity and gauge transformations in classical electromagnetism are summarized in Table 21.1.

A standard choice permits the linearized gravitational equations to be replaced by the two equations

$$\Box h_{\mu\nu}(x) = 0,$$

$$\partial_\nu h^\nu_\mu(x) - \tfrac{1}{2}\partial_\mu h^\nu_\nu(x) = 0,$$

where the first is the linearized Einstein equation and the second is a (Lorentz) gauge constraint, and where h^γ_ν is defined by

$$h^\gamma_\nu \equiv \eta^{\gamma\delta} h_{\delta\nu}$$

As in electromagnetism, the "Lorentz" gauge is really a family of gauges. We shall use that to simply things even further below.

21.2 Weak Gravitational Waves

Let us now seek solutions to

$$\Box h_{\mu\nu}(x) = 0,$$

$$\partial_\nu h^\nu_\mu(x) - \tfrac{1}{2}\partial_\mu h^\nu_\nu(x) = 0,$$

We expect the solution to be a superposition of components in the form

$$\begin{aligned} h_{\mu\nu}(x) &= \alpha_{\mu\nu} e^{ik\cdot x} + \alpha^*_{\mu\nu} e^{-ik\cdot x} \\ &= \alpha_{\mu\nu} e^{ik_\lambda x^\lambda} + \alpha^*_{\mu\nu} e^{-ik_\lambda x^\lambda} \end{aligned}$$

where $\alpha_{\mu\nu}$ is a symmetric 4×4 matrix called the *polarization tensor.*

21.2.1 States of Polarization

Since it is symmetric, the polarization tensor should generally have *10 independent components.* However, the demand that

$$\begin{aligned} h_{\mu\nu}(x) &= \alpha_{\mu\nu} e^{ik\cdot x} + \alpha^*_{\mu\nu} e^{-ik\cdot x} \\ &= \alpha_{\mu\nu} e^{ik_\lambda x^\lambda} + \alpha^*_{\mu\nu} e^{-ik_\lambda x^\lambda} \end{aligned}$$

satisfy the wave equation

$$\Box h_{\mu\nu}(x) = 0,$$

means that

$$k_\mu k^\mu = 0,$$

and the requirement that it satisfy the gauge condition

$$\partial_\nu h^\nu_\mu(x) - \tfrac{1}{2}\partial_\mu h^\nu_\nu(x) = 0,$$

implies that

$$k_\mu \alpha^\mu_\nu = \tfrac{1}{2} k_\nu \alpha^\mu_\mu,$$

where $k^\mu = \eta^{\mu\nu} k_\nu$ and so on (raise and lower indices with $\eta_{\mu\nu}$ in linearized gravity).

The four equations

$$k_\mu \alpha^\mu_\nu = \tfrac{1}{2} k_\nu \alpha^\mu_\mu,$$

reduce the number of independent components of α^ν_μ to six. But we have not yet exhausted the gauge (coordinate) degree of freedom because any coordinate transform

$$x^\mu \rightarrow x'^\mu = x^\mu + \varepsilon^\mu(x) \qquad h'_{\mu\nu} = h_{\mu\nu}(x) - \partial_\mu \varepsilon_\nu - \partial_\nu \varepsilon_\mu$$

that leaves

$$\partial_\nu h^\nu_\mu(x) - \tfrac{1}{2}\partial_\mu h^\nu_\nu(x) = 0,$$

valid does not alter the physical content of linearized gravity.

It is possible to use this freedom to set any four of the $h_{\mu\nu}$ to zero.

21.2.2 Polarization Tensor in Transverse–Traceless Gauge

We may use the freedom alluded to at the end of the preceding section to transform to *transverse traceless gauge (TT)* by choosing

$$h_{0i} = h_{ti} = 0 \qquad (i = 1, 2, 3)$$

$$\mathrm{Tr}\, h \equiv h^{\beta}_{\beta} = 0.$$

In terms of the polarization tensor $\alpha_{\mu\nu}$, preceding conditions correspond to

$$\alpha_{0i} = 0 \qquad \mathrm{Tr}\, \alpha = \alpha^{\beta}_{\beta} = 0.$$

The gauge conditions

$$\partial_\nu h^{\nu}_{\mu}(x) - \tfrac{1}{2}\partial_\mu h^{\nu}_{\nu}(x) = 0,$$

with $\mu = 0$ then require that $\alpha_{tt} \equiv \alpha_{00} = 0$ which, coupled with

$$\alpha_{0i} = 0$$

implies that four of the $\alpha_{\mu\nu}$ vanish:

$$\alpha_{0\mu} = 0.$$

Furthermore, for $i = 1, 2, 3$ the gauge conditions lead to the requirement that $ik^j \alpha_{ij} e^{ik\cdot x} = 0$, which is generally true only if the *transversality condition*,

$$k^j \alpha_{ij} = 0$$

is satisfied.

Let us take stock. We started with 10 independent components of the symmetric polarization tensor $\alpha_{\mu\nu}$. We then found that

- The condition $h_{0i} = 0$ requires $\alpha_{01} = \alpha_{02} = \alpha_{03} = 0$,
- The requirement

  $$\partial_\nu h^\nu_\mu(x) - \tfrac{1}{2}\partial_\mu h^\nu_\nu(x) = 0,$$

  yields $\alpha_{00} = 0$.
- Therefore, the four components $\alpha_{0\mu}$ of the symmetric polarization tensor vanish in TT gauge.
- The trace condition

  $$\mathrm{Tr}\, h \equiv h^\beta_\beta = 0.$$

  gives one constraint and the transversality condition

  $$k^j \alpha_{ij} = 0$$

  gives three additional ones for a total of four.

In TT gauge ten $\alpha_{\mu\nu}$, minus four $\alpha_{\mu\nu}$ that are identically zero, minus four constraints on $\alpha_{\mu\nu}$, leave *two independent physical polarizations for gravity waves.*

21.2.3 Helicity Components

As we have seen, there are six independent polarizations satisfying the wave equation

$$\Box h_{\mu\nu}(x) = 0,$$

and the general Lorentz gauge condition

$$\partial_\nu h^\nu_\mu(x) - \tfrac{1}{2}\partial_\mu h^\nu_\nu(x) = 0,$$

but the TT gauge shows explicitly that only two of these polarizations are physically meaningful. Further insight comes from asking how the $\alpha_{\mu\nu}$ change under rotations of the coordinate system about the z axis.

- One generally finds that a gravitational plane wave can be decomposed into helicity components ± 2, ± 1, 0, and 0, but
- The components with helicity 0 and ± 1 vanish under a suitable choice of coordinates. (Helicity is the projection of the angular momentum on the direction of motion.)
- Thus, only the helicity components ± 2 are physically relevant for gravitational waves, explaining why there are two independent physical states of polarization.

It is in this sense that we associate gravity with a spin-2 field.

Compare with the analogous situation in electromagnetism, which is described by a 4-vector field A^μ.

- This suggests that the Maxwell field should have 4 independent states of polarization α_μ.
- However, $k_\mu \alpha^\mu = 0$ reduces this to 3 and
- the freedom to make gauge transformations that leave the $\boldsymbol{E}$ and $\boldsymbol{B}$ fields unchanged demonstrates explicitly that the number of independent polarizations is actually only 2.
- Furthermore, a decomposition under rotations about the z axis yields helicities 0 and ± 1, but only the helicities ± 1 are physically relevant.
- These correspond to the 2 independent states of polarization for a massless vector (spin-1) field.

That there are only two physical states of polarization for the photon is tied intimately to its masslessness and associated local gauge invariance.

- A *massive* vector field has 3 states of polarization and is not locally gauge invariant.
- Likewise, that the gravitational field exhibits only 2 physical states of polarization is a consequence of the masslessness of the graviton.
- A massive spin-2 field would have 5 physical polarization states.

21.2.4 General Solution in Transverse–Traceless Gauge

To display polarizations explicitly, we assume the gravitational wave to propagate on the z axis with energy ω (in $\hbar = c = 1$ units). We have

$$k_\mu k^\mu = 0,$$

and the 4-momentum vector must have the form

$$k^\mu = (\omega, 0, 0, \omega)$$

The transversality condition $k^j \alpha_{ij} = 0$ is explicitly

$$\begin{aligned} k^1\alpha_{11} + k^2\alpha_{12} + k^3\alpha_{13} &= 0 \\ k^1\alpha_{21} + k^2\alpha_{22} + k^3\alpha_{23} &= 0 \\ k^1\alpha_{31} + k^2\alpha_{32} + k^3\alpha_{33} &= 0. \end{aligned}$$

But from $k^\mu = (\omega, 0, 0, \omega)$, we have $k^1 = k^2 = 0$, so

$$\alpha_{13} = \alpha_{23} = \alpha_{33} = 0,$$

and from earlier $\alpha_{0\mu} = 0$. Therefore, for the symmetric matrix $\alpha_{\mu\nu}$ the only nonvanishing components are α_{11}, $\alpha_{12} = \alpha_{21}$, and α_{22}, and these are further constrained by the trace requirement

$$\mathrm{Tr}\,\alpha = \alpha_\beta^\beta = 0,$$

so $\alpha_{11} = -\alpha_{22}$.

Finally, from

$$k^\mu = (\omega, 0, 0, \omega)$$

we have

$$ik \cdot x = -i(\omega t - \omega z)$$

and the general solution of the linearized Einstein equations for z axis propagation with fixed frequency ω in transverse–traceless gauge is

$$h_{\mu\nu}(t,z) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & \alpha_{11} & \alpha_{12} & 0 \\ 0 & \alpha_{12} & -\alpha_{11} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} e^{i\omega(z-t)},$$

which exhibits explicitly the transverse and traceless properties, with two independent polarization states.

- The part of the wave that is proportional to $\alpha_{xx} = \alpha_{11}$ is called the *plus polarization* (denoted by $+$) and
- the part proportional to $\alpha_{xy} = \alpha_{12} = \alpha_{21}$ is called the *cross polarization* (denoted by $\times$).
- For example a purely cross-polarized plane wave propagating in the z direction may be represented as

$$h_{\mu\nu}(t,z) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} e^{i\omega(z-t)}.$$

- The most general gravitational wave is a superposition of waves having the form

$$h_{\mu\nu}(t,z) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & \alpha_{11} & \alpha_{12} & 0 \\ 0 & \alpha_{12} & -\alpha_{11} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} e^{i\omega(z-t)},$$

  with different ω, directions of propagation, and amplitudes for the two polarizations.

- In linearized approximation and TT gauge, it may be expressed as

$$h_{\mu\nu}(t,z) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & f_+(t-z) & f_\times(t-z) & 0 \\ 0 & f_\times(t-z) & -f_+(t-z) & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

  assuming propagation of the gravitational wave along the z axis.

21.3 Response of Test Particles to Gravitational Waves

We cannot detect a gravitational wave locally

- In a local enough region the effects of gravity may be transformed away *(equivalence principle).*
- Thus the effect of a gravitational wave on a single point test particle has *no measurable consequences.*

Gravitational waves may be detected only by their influence on *two or more test particle at different locations.*

21.3.1 Response of Two Test Masses

- Assume a linearized gravitational wave of $+$ polarization propagating on the z axis,

$$h_{\mu\nu}(t,z) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} f(t-z),$$

  with a corresponding time-dependent metric

$$ds^2 = -dt^2 + (1 + f(t-z))dx^2 + (1 - f(t-z))dy^2 + dz^2.$$

- Consider two test masses with
  - mass A initially at rest at the origin, $x^i_{\mathrm{A}} = (0,0,0)$,
  - mass B initially at rest at a point $x^i_{\mathrm{B}} = (x_{\mathrm{B}}, y_{\mathrm{B}}, z_{\mathrm{B}})$, and
  - a gravitational wave propagating on the z axis.

- Since the particles are at rest before the gravitational wave arrives, the initial 4-velocities are

$$u_{\mathrm{A}} = u_{\mathrm{B}} = (1,0,0,0).$$

- For the test masses the geodesic equation is given by

$$\frac{d^2x^i}{d\tau^2} = -\Gamma^i_{\mu\nu} u^\mu u^\nu = -\Gamma^i_{\mu\nu} \frac{dx^\mu}{d\tau} \frac{dx^\nu}{d\tau}.$$

The undisturbed spacetime is assumed flat and $\Gamma^i_{\mu\nu}$ vanishes.

- From

$$\delta\Gamma^\gamma_{\mu\nu} = \frac{1}{2}\eta^{\gamma\delta}\left(\frac{\partial h_{\delta\mu}}{\partial x^\nu} + \frac{\partial h_{\delta\nu}}{\partial x^\mu} - \frac{\partial h_{\mu\nu}}{\partial x^\delta}\right).$$

  we have to first order

$$\begin{aligned}\frac{d^2(\delta x^i)}{d\tau^2} &= -\delta\Gamma^i_{\mu\nu}u^\mu u^\nu = -\delta\Gamma^i_{00} \\ &= -\frac{1}{2}\eta^{i\delta}\left(\frac{\partial h_{\delta 0}}{\partial x^0} + \frac{\partial h_{\delta 0}}{\partial x^0} - \frac{\partial h_{00}}{\partial x^\delta}\right),\end{aligned}$$

  where

$$u_{\mathrm{A}} = u_{\mathrm{B}} = (1,0,0,0).$$

  has been used.

- But in TT gauge $h_{\delta 0} = 0$, so

$$\frac{d^2(\delta x^i)}{d\tau^2} = -\delta\Gamma^i_{00} = 0$$

To this order, the *coordinate distance* between A and B is *not changed* by the gravitational wave.

However, the *proper distance* between A and B is changed.

For example, assume A and B to lie on the x axis and to be separated by a distance L_0. From

$$ds^2 = -dt^2 + (1 + f(t-z))dx^2 + (1 - f(t-z))dy^2 + dz^2.$$

the relevant line element is then

$$ds^2 = -dt^2 + (1 + f(t-z))dx^2,$$

corresponding to the metric

$$g_{\mu\nu} = \begin{pmatrix} -1 & 0 \\ 0 & 1 + h_{xx}(t,0) \end{pmatrix},$$

where $h_{xx} = h_{11}$ and we have chosen $z = 0$ at time t.

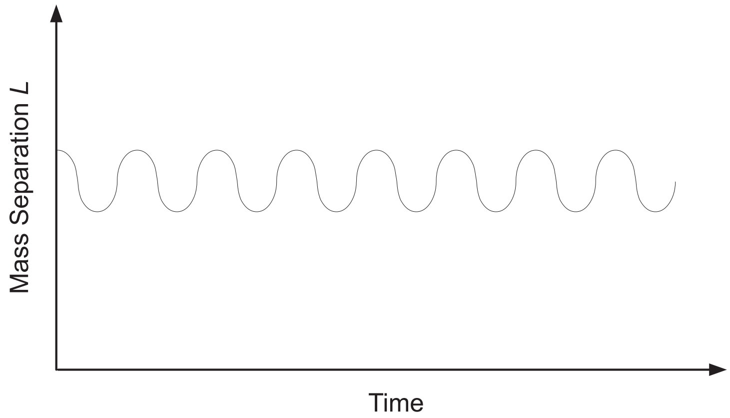


Figure 21.2: Test mass separation perturbed by a gravitational wave.

Then the proper distance between A and B is

$$
\begin{aligned}
L(t) &= \int_0^{L_0} (-\det g)^{1/2}\, dx \\
&= \int_0^{L_0} (1 + h_{xx}(t,0))^{1/2}\, dx \\
&\simeq \int_0^{L_0} (1 + \tfrac{1}{2} h_{xx}(t,0))\, dx = \left(1 + \tfrac{1}{2} h_{xx}(t,0)\right) L_0 .
\end{aligned}
$$

Therefore the change in distance between the masses is

$$
\frac{\delta L(t)}{L_0} \simeq \tfrac{1}{2} h_{xx}(t,0),
$$

which oscillates with the time dependence of the gravitational wave, as indicated schematically in Fig. 21.2.

> The amplitude of the variation is greatly exaggerated in Fig. 21.2, however! One expects that $\delta L / L_0 \sim 10^{-21}$ for a typical fluctuation associated with gravitational waves detectable on Earth from distant astronomical events.

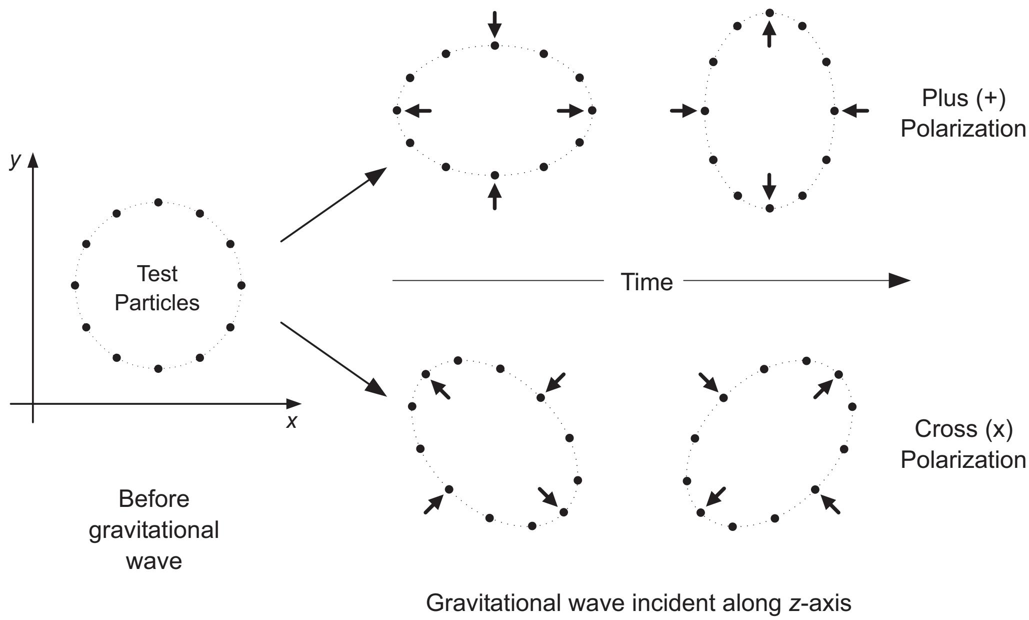


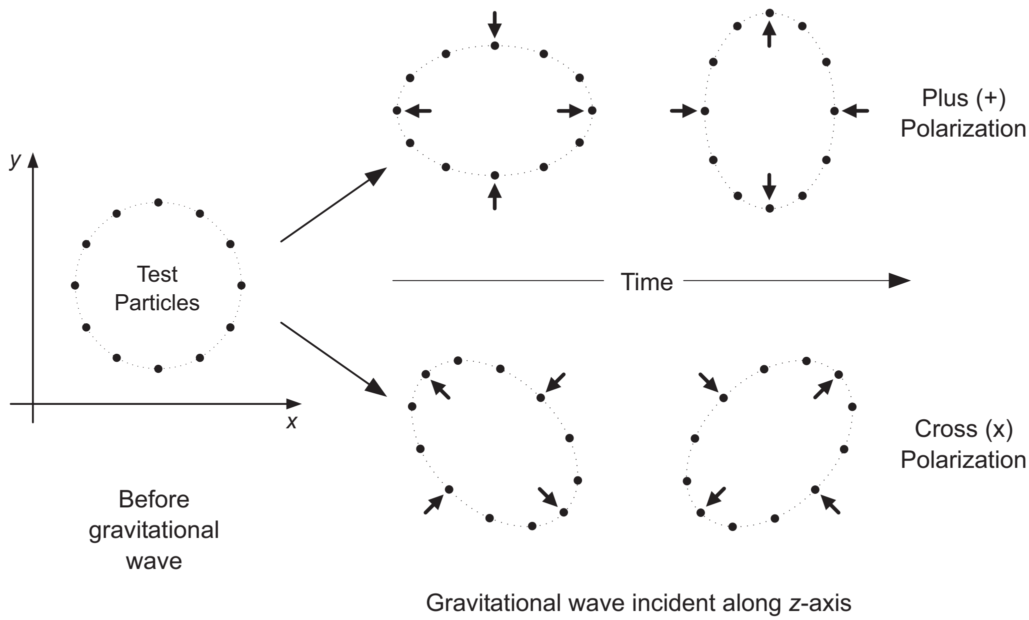
Figure 21.3: Effect of a gravitational wave incident along the z axis on an array of test masses in the x–y plane.

21.3.2 The Effect of Polarization

The effect of the gravitational wave polarization may be illustrated by the effect of a gravitational wave on an planar test mass array.

- Like electromagnetic waves, gravitational waves are transverse; thus only separations in the transverse directions (x and y, if the wave is incident along z) are changed by the gravitational wave.
- The effect of purely plus-polarized and purely cross-polarized gravitational waves on an initially circular array of test masses is illustrated in Fig. 21.3.

As in Fig. 21.2, the magnitude of the effect is *greatly exaggerated* relative to that of gravitational waves detectable on Earth from distant sources.



Each polarization is seen to give rise to an elliptical oscillation in the distribution of the test masses, with the cross polarization ellipse rotated by $\frac{\pi}{4}$ relative to the corresponding plus polarization ellipse.

- The 45^{o} relative rotation results from gravity being described by a rank-2 tensor field (spin-2 field).
- In electromagnetism the fields correspond to a rank-1 tensor A^{μ} (spin-1 or vector field) and the rotation angle between the two independent states of polarization is instead 90^{o}.

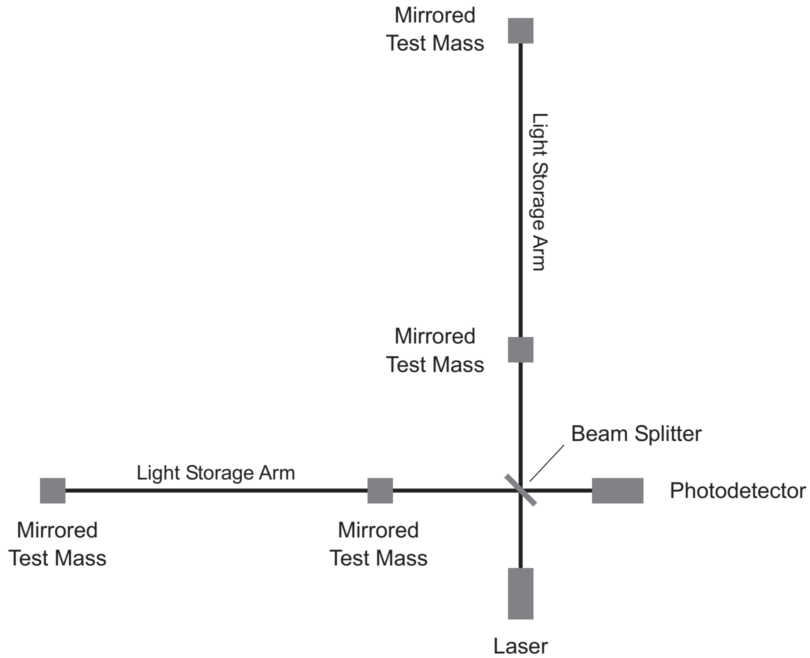


Figure 21.4: Schematic laser interferometer gravity wave detector. In the light storage arms light is multiply reflected, increasing the effective length of the arms.

21.4 Gravitational Wave Detectors

Gravity wave detectors tend to use Michelson laser interferometers with kilometer or longer arms. A typical implementation is illustrated in Fig. 21.4.

- Laser light is split and directed down two arms.
- Suspended, mirrored test masses reflect the light at the ends of the arms.
- The reflected light is recombined and interference fringes are analyzed for evidence indicating changes in the distances to the test masses.

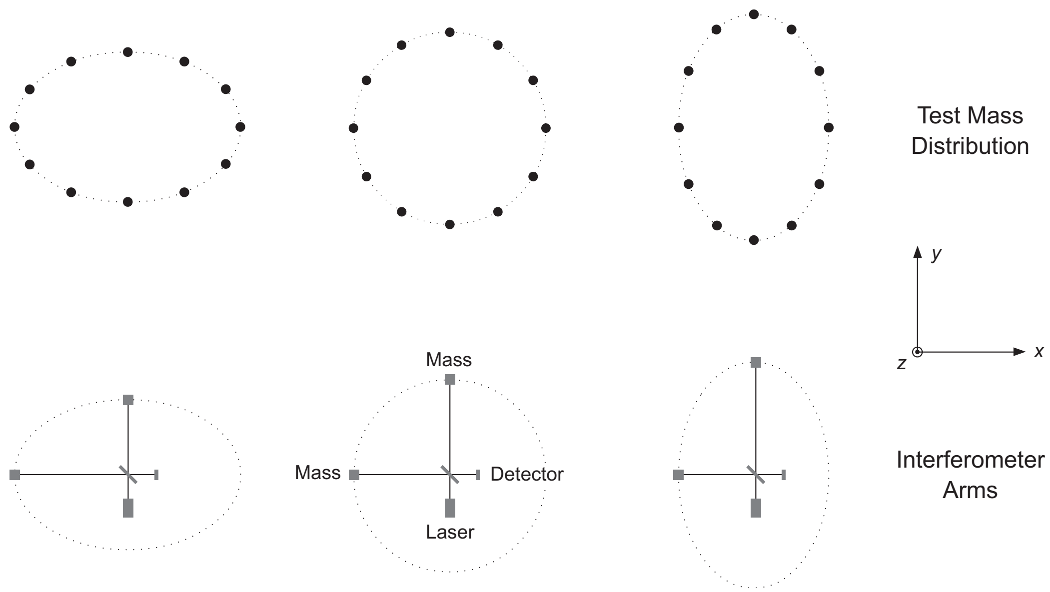


Figure 21.5: Analogy between interaction of a gravitational wave with a test mass distribution and with an interferometer.

Because one is interfering light over long path lengths, it is possible to detect extremely small changes in distance.

- This is necessary, since to detect gravitational waves from merging neutron stars or core-collapse supernovae, fractional changes in distance of order 10^{-21} or smaller must be measured.
- 10^{-21} is approximately the ratio of the width of a human hair to the distance to Alpha Centauri!

Laser interferometers may be viewed as extremely precise ways to measure the distortions illustrated in Fig. 21.3 for a small number of test masses (Fig. 21.5).

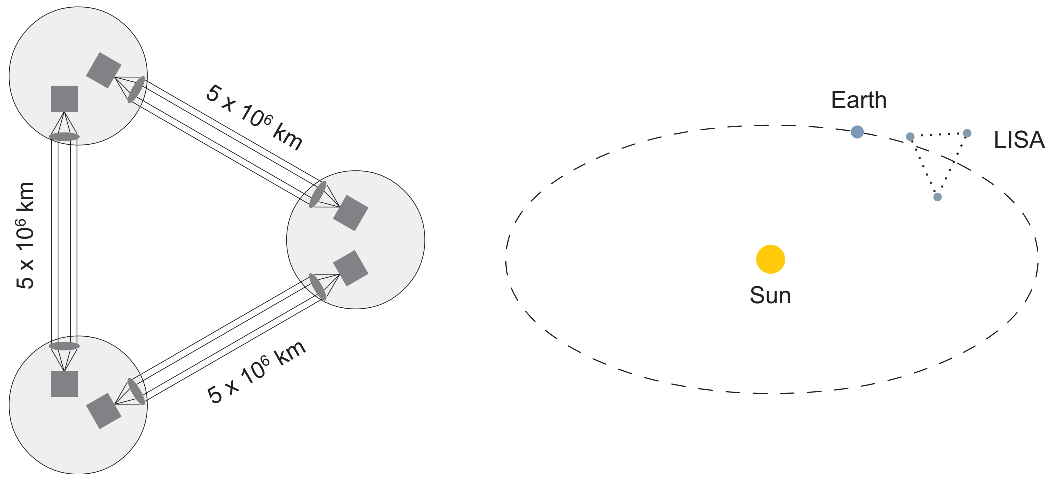


Figure 21.6: The proposed LISA space-based gravitational wave antenna.

Ground-based detectors such as LIGO are now beginning to operate. A proposed space-based array, LISA, would have 5 million kilometer interferometer arms and could be launched within a decade. The schematic arrangement for LISA, and its proposed orbit, are illustrated in Fig. 21.6.

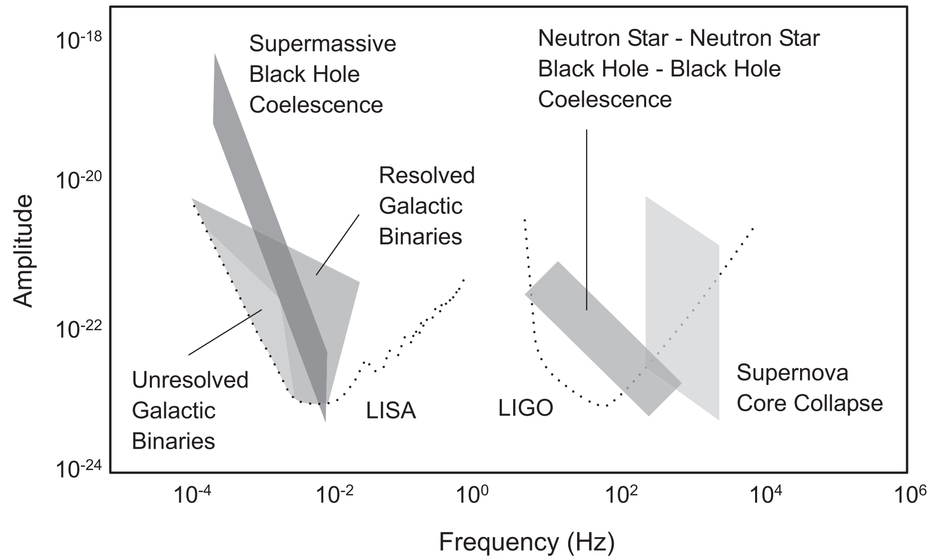


Figure 21.7: Amplitude and frequency ranges expected for gravitational waves from various sources. The lower detection ranges for LIGO and LISA are indicated by dotted lines.

The amplitude and frequency ranges for LIGO and LISA, along with ranges expected for important astrophysical sources of gravitational waves that may be observable, are illustrated in Fig. 21.7. Space-based interferometers like LISA can go to much lower frequencies because they can have very long interferometer arms and because there is little interference from environmental noise.

The preceding discussion introduced the basic idea of gravitational waves in terms of a linearized approximation to gravity.

- In the following we consider potential sources of gravitational waves that might be detectable by Earth-based or space-based gravitational wave detectors.
- We begin with the simpler case of weak gravitational waves.

We shall conclude with a general introduction to the more difficult problem of describing strong sources of gravitational waves.

21.5 Production of Weak Gravitational Waves

To study the production of gravitational waves, we must in general solve the full non-linear Einstein equations with source terms $T_{\mu\nu}$.

- This is a formidable problem, generally only tractable for large-scale computation (*numerical relativity*).
- However, we can gain considerable insight by studying a less complex situation, the linearized Einstein equation with sources.
- This is an analytically accessible problem that has many parallels with the study of sources for electromagnetic waves.
- It has been shown by numerical simulation that many key features for the production of weak gravitational waves carry over in recognizable form for the production of gravitational waves in strong-gravity environments.
- Therefore, our approach will be to concentrate on a more quantitative treatment of sources for weak gravitational waves

We shall then conclude with qualitative order–of–magnitude remarks and some numerically computed examples for strong-gravity wave sources.

21.5.1 Energy Densities

In an electromagnetic field or a Newtonian gravitational field it makes sense to talk about local energy densities.

- For example, in a local Newtonian field the energy density at a point $\boldsymbol{x}$ is given by

$$\varepsilon(\boldsymbol{x}) = -\frac{1}{8\pi G}(\nabla\Phi(\boldsymbol{x}))^2,$$

  where $\Phi(\boldsymbol{x})$ is the Newtonian gravitational potential.

- There is *no corresponding local energy density* in general relativity.

- If such a density existed, it would *contradict the equivalence principle,* which requires gravity to *vanish* in a sufficiently local region (local inertial frame).

- However, it *does* make sense to speak of an approximate energy density associated with a weak gravitational wave of wavelength λ, provided that λ *is much shorter than the curvature* R *of the background spacetime* through which the wave propagates.

Such approximations become very good at *large distances from the source* of a gravitational wave, where curvature associated with the source becomes negligible and $\lambda/R \to 0$.

Therefore, we may formulate a description of energy loss from gravitational wave sources at large distances from the source where we may associate an *approximate energy density* with a wave by averaging over several wavelengths.

21.5.2 Multipolarities

The lowest-order contribution from a source to electromagnetic radiation corresponds to *dipole motion of the source.*

- The gravitational field is a *tensor rather than vector field.*
- Like the electromagnetic field, the production of gravitational waves requires *non-spherical motion of the charge* (which is electrical charge for the electromagnetic field and inertial mass for the gravitational field).
- However, for the gravitational field no monopole or dipole component contributes to the generation of gravitational waves.
- The lowest order gravitational wave generation that is permitted corresponds to time-dependent *quadrupole distortions of the source mass.*

As a result, many of the formulas for sources of electromagnetic waves and for weak gravitational waves are similar, but not identical.

21.5.3 Linearized Einstein Equation with Sources

The metric perturbation for long-wavelength gravitational waves far from a non-relativistic source (wavelengths much larger than the characteristic source size imply low velocities for mass in the source relative to that of light), is

$$\bar{h}^{ij}(t,\boldsymbol{x})_{r\to\infty} \simeq \frac{2}{r}\ddot{I}^{ij}(t-r),$$

where

- double dots denote the second time derivative and
- $\bar{h}^{ij}(t,\boldsymbol{x})$ is the *trace-reversed amplitude,*

$$\begin{aligned}\bar{h}_{\mu\nu}(t,\boldsymbol{x}) &\equiv h_{\mu\nu} - \tfrac{1}{2}\eta_{\mu\nu}h^{\gamma}_{\gamma} \\ &= h_{\mu\nu} - \tfrac{1}{2}\eta_{\mu\nu}\mathrm{Tr}\,h,\end{aligned}$$

  which satisfies

$$\Box\bar{h}_{\mu\nu} = -16\pi T_{\mu\nu}.$$

The *second mass moment* $I^{ij}(t)$ is given by

$$I^{ij}(t) \equiv \int \rho(t,\boldsymbol{x})x^i x^j d^3x\,,$$

where $\rho(t,\boldsymbol{x})$ is the mass density of the source.

The stress–energy tensor $T_{\mu\nu}$ appearing in the linearized relation

$$\Box \bar{h}_{\mu\nu} = -16\pi T_{\mu\nu}.$$

has non-zero components of the form

$$T^{00} = \frac{T^{03}}{c} = \frac{T^{33}}{c} = \frac{c^2}{16\pi G}\left\langle (\dot{a}_+)^2 + (\dot{a}_\times)^2 \right\rangle,$$

where

- $\langle \dots \rangle$ denotes an average over several wavelengths,
- a_+ and $a_\times$ denote contributions from the two possible polarizations,
- T^{00} is the energy density,
- T^{03} is the energy flux (c^2 times the momentum density),
- T^{33} is the momentum flux.

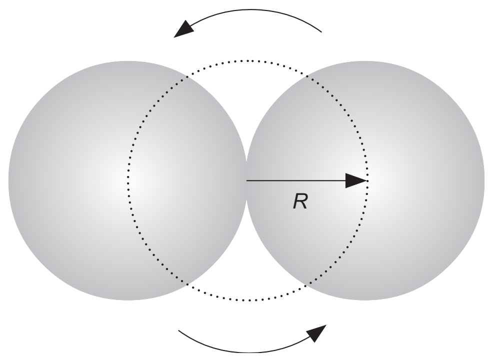


Figure 21.8: A contact binary as a source of gravitational waves.

21.5.4 Gravity Wave Amplitudes

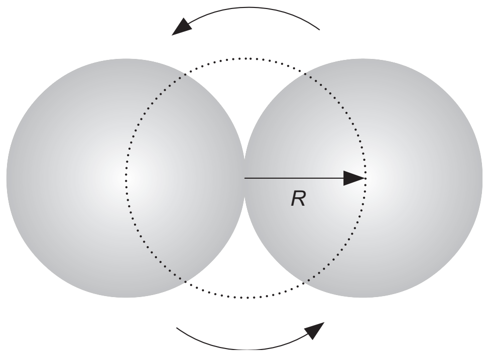
The gravity wave amplitude $\bar{h}^{ij}$ is given by

$$\bar{h}^{ij}(t,\boldsymbol{x})_{r\to\infty} \simeq \frac{2}{r}\ddot{I}^{ij}(t-r),$$

in linear approximation.

- Let us make some estimates based on this formula using as a simple model for gravitational wave emission a binary star system in an orbit such that the surfaces of the two stars touch (contact binary).
- For simplicity, we shall assume the two stars to be of the same radius and mass, and to revolve in circular orbits about their center of mass.

Figure 21.8 illustrates.



The second mass moment is

$$I^{ij}(t) = \int \rho(t, \boldsymbol{x}) x^i x^j d^3x = 2MR^2.$$

The system revolves with a period P and taking the derivative twice with respect to time gives a factor $1/P^2$

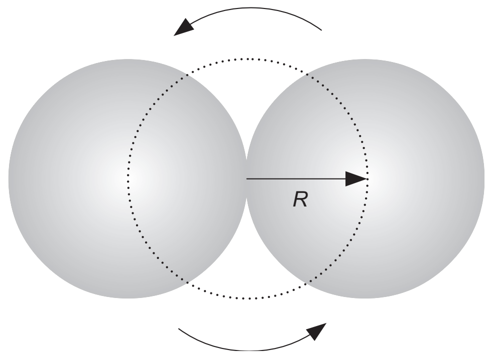
$$\ddot{I}^{ij} \simeq 2\frac{MR^2}{P^2}.$$

Insertion of this approximation in

$$\bar{h}^{ij}(t, \boldsymbol{x})_{r\to\infty} \simeq \frac{2}{r}\ddot{I}^{ij}(t-r),$$

leads to

$$\bar{h}^{ij} \simeq \frac{2}{r}\ddot{I}^{ij} = \frac{4MR^2}{rP^2}.$$



The relation

$$\bar{h}^{ij} \simeq \frac{2}{r}\ddot{I}^{ij} = \frac{4MR^2}{rP^2}.$$

can be expressed in terms of source velocities by noting that

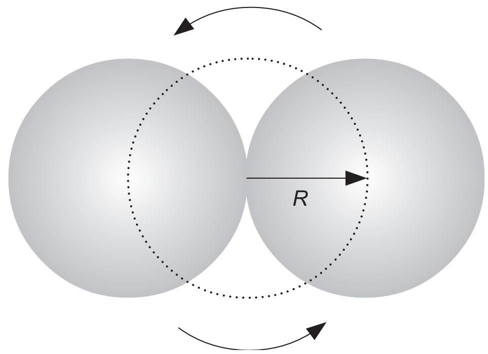
- In one period the center of mass for each star travels a distance $C = 2\pi R$.
- Therefore, the velocity of the center of mass for a star is

$$v = \frac{2\pi R}{P},$$

- This can be solved for the ratio R/P and used to rewrite $\bar{h}^{ij}$ as

$$\bar{h}^{ij} \simeq \frac{4MR^2}{rP^2} \quad \longrightarrow \quad \bar{h}^{ij} \simeq \frac{Mv^2}{\pi^2 r}.$$

This is a specialized result obtained assuming crude approximations, but more careful derivations indicate that it has broader validity than its derivation would suggest.



If we

- Note that the Schwarzschild radius and mass are related by $r_S = 2M$,
- reinsert factors of c and G, and
- drop the numerical factors (note that the numerical factors dropped are larger than order unity),

the expression

$$\bar{h}^{ij} \simeq \frac{Mv^2}{\pi^2 r}.$$

may be rewritten as

$$\bar{h} \simeq \frac{r_S}{r}\frac{v^2}{c^2}.$$

> The amplitude of the metric perturbation associated with the gravitational wave is largest for compact sources that have radii comparable with their Schwarzschild radii.

As noted above, the result

$$\bar{h} \simeq \frac{r_S}{r} \frac{v^2}{c^2}.$$

is more general than its derivation might suggest and we shall take it as a qualitative guide to the amplitude of gravitational waves far from a weak source.

Weak gravitational waves are generated primarily by systems that are gravitationally bound or nearly so.

- Therefore *the virial theorem is applicable* and to order of magnitude we may expect that the kinetic and potential energies are comparable, which implies

$$\tfrac{1}{2}Mv^2 \sim \frac{GM^2}{R},$$

- From this we may write

$$\frac{v^2}{c^2} \simeq \frac{2GM^2}{MRc^2} = \frac{r_S}{R} = \varepsilon^{2/7},$$

  where we have defined a gravitational wave emission efficiency factor

$$\varepsilon \equiv \left(\frac{r_S}{R}\right)^{7/2}$$

  (the justification for terming this the gravitational wave efficiency will be given below).

Therefore, $\bar{h}$ may be expressed in the form

$$\bar{h} \simeq \frac{r_S}{r}\frac{v^2}{c^2} \simeq \frac{r_S^2}{rR} = \varepsilon^{2/7}\frac{r_S}{r},$$

$$= 9.55 \times 10^{-17}\varepsilon^{2/7}\left(\frac{M}{M_\odot}\right)\left(\frac{\text{kpc}}{r}\right),$$

which is dimensionless.

21.5.5 Amplitudes and Event Rates

We may use the preceding results to estimate amplitudes and corresponding event rates for candidate gravitational wave events that might be detectable by the current generation of gravitational wave interferometers.

- For estimation purposes, let us assume that $\varepsilon = (r_{\mathrm{s}}/R)^{7/2} \sim 0.1$ and that the mass participating in gravitational wave generation is $M \sim M_\odot$
- (these would be reasonable guesses for merging neutron stars or asymmetric core collapse supernovae, which are expected to be two classes of events with high probability for observation by LIGO).
- If we first consider events within the galaxy, we may assume that on average $r \sim 10$ kpc.
- Then we may estimate

$$\begin{aligned}\bar{h} &= 9.55 \times 10^{-17} \varepsilon^{2/7} \left(\frac{M}{M_\odot}\right) \left(\frac{\mathrm{kpc}}{r}\right) \\ &\simeq 5 \times 10^{-18}.\end{aligned}$$

- But such events occur within the galaxy only *once every 50 years or so,* on average.

Therefore, to obtain a reasonable event rate we must look for gravitational waves from sources at larger distances.

The nearest rich cluster of galaxies is Virgo, at a distance of about 15 Mpc.

- Therefore, if we go out to 15 Mpc average distance, we may expect event rates for gravitational waves from merging neutron stars and asymmetric core-collapse supernovae to go up to *ten or more per year* because we are now surveying thousands of galaxies.
- But the average observed metric perturbation (which is directly related to the strain detected by the detectors) becomes $\bar{h} \sim 3 \times 10^{-21}$.

Thus, we expect that to detect systematic gravitational wave events the detectors must be able to sample strains reliably at the $\Delta L/L_0 \simeq 10^{-21}$ level or better.

21.5.6 Power in Gravitational Waves

The power radiated in gravitational waves for a system that has velocities well below c and weak internal gravity is given by the *quadrupole formula*,

$$L = \frac{dE}{dt} = \frac{1}{5}\left\langle \dddot{I\!\!\!-}_{ij}\dddot{I\!\!\!-}^{ij}\right\rangle = \frac{1}{5}\frac{G}{c^5}\left\langle \dddot{I\!\!\!-}_{ij}\dddot{I\!\!\!-}^{ij}\right\rangle,$$

- $\langle\ \rangle$ denotes a time average over a period,
- the triple dot means a third time derivative, and
- the reduced quadrupole tensor $I\!\!\!-$ is defined by

$$I\!\!\!-^{ij} \equiv I^{ij} - \tfrac{1}{3}\delta^{ij}\mathrm{Tr}\, I,$$

where $\mathrm{Tr}\, I \equiv I^k_k$.

> This formula is the gravitational analog of the formula for radiated power in electromagnetism but has a different factor (1/5 instead of 1/20) and corresponds to quadrupole rather than dipole radiation.

As shown in an Exercise,

$$L = \frac{dE}{dt} = \frac{1}{5}\frac{G}{c^5}\left\langle \dddot{I}_{ij}\dddot{I}^{ij} \right\rangle,$$

may be reduced to

$$L \simeq L_0 \frac{r_S^2}{R^2}\left(\frac{v}{c}\right)^6,$$

where the scale for radiated gravitational wave power is set by

$$L_0 \equiv \frac{c^5}{G} = 3.6 \times 10^{59} \text{ erg s}^{-1},$$

and the total energy ΔE emitted in one period P can then be calculated as

$$\Delta E \simeq LP \simeq Mc^2\left(\frac{r_S}{R}\right)^{7/2} = \varepsilon Mc^2.$$

Therefore, we conclude that

$$\varepsilon = \left(\frac{r_S}{R}\right)^{7/2}$$

parameterizes the *efficiency of gravitational wave emission* by the mass M.

21.6 Gravitational Radiation from Binary Systems

In preceding sections we have made some qualitative estimates for gravitational wave emission from binary stars. In this section we derive in a somewhat more rigorous fashion a formalism applicable for such systems.

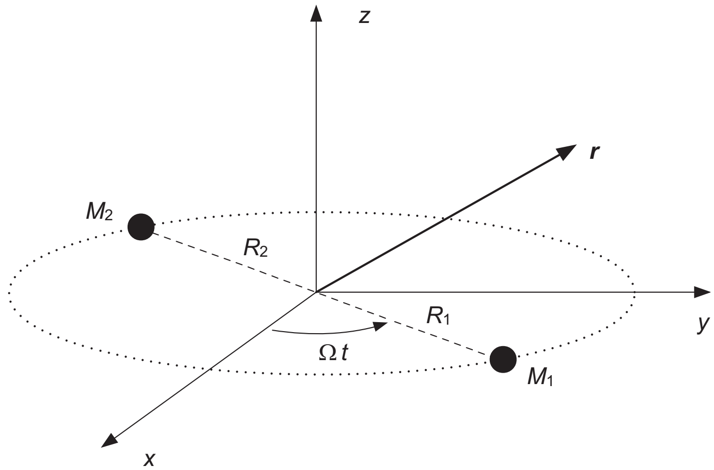


Figure 21.9: Coordinate system for binary stars in circular orbits.

21.6.1 Gravitational Wave Luminosity

A binary system is illustrated in Fig. 21.9. For simplicity we assume

- circular orbits
- $M_1 = M_2 \equiv M$, and therefore that $R_1 = R_2 \equiv R$.

Introducing polar coordinates for the center of mass for the first mass,

$$x(t) = R\cos\Omega t \qquad y(t) = R\sin\Omega t \qquad z(t) = 0,$$

the components of the second mass moment are given by

$$I^{ij}(t) \equiv \int \rho(t, \boldsymbol{x}) x^i x^j d^3x\,,$$

which reduces to

$$I^{ij} = 2Mx^i(t)\,x^j(t).$$

for two discrete masses.

Explicitly, the non-zero components are

$$
\begin{aligned}
I^{11} &= I^{xx} = 2MR^2\cos^2\Omega t = MR^2(1+\cos 2\Omega t)\\
I^{12} &= I^{xy} = 2MR^2\cos\Omega t\sin\Omega t = MR^2\sin 2\Omega t\\
I^{22} &= I^{yy} = 2MR^2\sin^2\Omega t = MR^2(1-\cos 2\Omega t)
\end{aligned}
$$

The trace-reversed amplitude is given by

$$\bar{h}^{ij}(t,\boldsymbol{x})_{r\to\infty} \simeq \frac{2}{r}\ddot{I}^{ij}(t-r),$$

which requires the second time derivatives of the I^{ij}. These are easily computed. For example,

$$
\begin{aligned}
\dot{I}^{xx}(t) &= \frac{d}{dt}\Big(MR^2(1+\cos 2\Omega t)\Big) = -2\Omega MR^2\sin 2\Omega t\\
\ddot{I}^{xx}(t) &= -4\Omega^2MR^2\cos 2\Omega t\\
\bar{h}^{xx}_{r\to\infty} &= \frac{2}{r}\ddot{I}^{xx}(t-r) = \frac{-8\Omega^2MR^2}{r}\cos 2\Omega(t-r).
\end{aligned}
$$

Computing the time derivatives for the other components in like manner we obtain

$$\bar{h}^{ij}_{r\to\infty} = \frac{8\Omega^2MR^2}{r}\begin{pmatrix} -\cos 2\Omega(t-r) & -\sin 2\Omega(t-r) & 0\\ -\sin 2\Omega(t-r) & \cos 2\Omega(t-r) & 0\\ 0 & 0 & 0\end{pmatrix}.$$

Note that the appearance of 2Ω in the arguments of the above equation means that the frequency ω of the gravitational wave radiation will be twice the rotational frequency Ω.

The reduced moment $\bar{I}^{ij}$ is given by

$$\bar{I}^{ij} \equiv I^{ij} - \tfrac{1}{3}\delta^{ij}\mathrm{Tr}\, I,$$

but from

$$I^{xx} = 2MR^2\cos^2\Omega t = MR^2(1+\cos 2\Omega t)$$
$$I^{yy} = 2MR^2\sin^2\Omega t = MR^2(1-\cos 2\Omega t)$$

the trace,

$$\mathrm{Tr}\, I = I^{xx} + I^{yy} = MR^2,$$

is independent of time. Therefore, $\dot{\bar{I}}^{ij} = \dot{I}^{ij}$ and the radiated luminosity is

$$L = \frac{1}{5}\left\langle \dddot{\bar{I}}_{ij}\dddot{\bar{I}}^{ij}\right\rangle = \frac{1}{5}\left\langle \dddot{I}_{ij}\dddot{I}^{ij}\right\rangle.$$

From

$$\ddot{I}^{xx}(t) = -4\Omega^2 MR^2\cos 2\Omega t$$

and the corresponding expressions for the other components, the triple time derivatives are

$$\dddot{I}^{xx}(t) = 8\Omega^3 MR^2\sin 2\Omega t$$
$$\dddot{I}^{xy}(t) = \dddot{I}^{yx}(t) = -8\Omega^3 MR^2\cos 2\Omega t$$
$$\dddot{I}^{yy}(t) = -8\Omega^3 MR^2\sin 2\Omega t$$

Thus, we have

$$\begin{aligned} L &= \frac{1}{5}\left\langle \dddot{I}_{ij}\dddot{I}^{ij}\right\rangle \\ &= \frac{1}{5}\left\langle (\dddot{I}^{xx})^2 + 2(\dddot{I}^{xy})^2 + (\dddot{I}^{yy})^2\right\rangle \\ &= \frac{1}{5}\left[\frac{1}{P}\int_0^P \left((\dddot{I}^{xx})^2 + 2(\dddot{I}^{xy})^2 + (\dddot{I}^{yy})^2\right) dt\right] \\ &= \frac{64\Omega^6 M^2 R^4}{5P}\int_0^P \left(\sin^2 2\Omega t + 2\cos^2 2\Omega t + \sin^2 2\Omega t\right) dt \\ &= \frac{128}{5}\Omega^6 M^2 R^4. \end{aligned}$$

The frequency Ω and the period P are related by $\Omega = 2\pi/P$ and from Kepler's third law,

$$R = \left(\frac{MP^2}{16\pi^2}\right)^{1/3},$$

so the gravitational wave luminosity may also be expressed in terms of the mass and period as

$$\begin{aligned} L &= \frac{128}{5}4^{1/3}\left(\frac{\pi M}{P}\right)^{10/3} = \frac{128}{5}4^{1/3}\frac{c^5}{G}\left(\frac{\pi G M}{c^3 P}\right)^{10/3} \\ &= 1.9\times 10^{33}\left(\frac{M}{1M_\odot}\right)^{10/3}\left(\frac{1\mathrm{h}}{P}\right)^{10/3} \ \mathrm{erg\ s^{-1}}. \end{aligned}$$

These formulas have assumed circular orbits and equal masses for the components of the binary. For generalizations to binary systems for which $M_1 \neq M_2$ and non-zero eccentricity, see Shapiro and Teukolsky §16.4.

21.6.2 Influence of Gravitational Radiation on Binary Orbit

In Newtonian approximation the total energy of a binary is

$$E = E_1^{\text{kinetic}} + E_2^{\text{kinetic}} + E^{\text{grav}} = Mv^2 - \frac{M^2}{2R}.$$

But from $v = 2\pi R/P$ and Kepler's third law,

$$Mv^2 = M\left(\frac{4\pi^2 R^2}{P^2}\right) \qquad R = \left(\frac{MP^2}{16\pi^2}\right)^{1/3},$$

which may be used to rewrite the total energy as

$$E = -\frac{M^2}{4R} = -\frac{M}{4}\left(\frac{4\pi M}{P}\right)^{2/3}.$$

The total energy is negative (it is a bound system), so if the energy is reduced by gravitational wave emission the period P must *decrease.*

Thus, gravitational wave emission tends to *decrease the period* (and thus the size) of a binary orbit. Although we will not prove it, the emission of gravitational wave radiation also tends to *circularize elliptical orbits.*

Differentiating

$$E = -\frac{M^2}{4R} = -\frac{M}{4}\left(\frac{4\pi M}{P}\right)^{2/3}.$$

with respect to t assuming the mass to be constant, equating to the negative of the gravitational wave luminosity ,

$$L = \frac{128}{5}4^{1/3}\left(\frac{\pi M}{P}\right)^{10/3} = \frac{128}{5}4^{1/3}\frac{c^5}{G}\left(\frac{\pi GM}{c^3 P}\right)^{10/3}$$
$$= 1.9\times 10^{33}\left(\frac{M}{1M_\odot}\right)^{10/3}\left(\frac{1\text{h}}{P}\right)^{10/3}\ \text{erg s}^{-1}.$$

and solving the resulting eqation for dP/dt gives

$$\frac{dP}{dt} = -\frac{96\pi}{5}4^{1/3}\left(\frac{2\pi M}{P}\right)^{5/3}$$
$$= -3.4\times 10^{-12}\left(\frac{M}{M_\odot}\right)^{5/3}\left(\frac{1\ \text{h}}{P}\right)^{5/3}$$

This is a dimensionless measure of how the orbital period is altered over time by the emission of gravitational wave radiation from the binary system.

21.6.3 Gravitational Wave Emission from the Binary Pulsar

The Binary Pulsar consists of a pulsar and a compact companion (most likely also a neutron star) in orbit around their common center of mass.

- Each star has a mass near 1.4 $M_\odot$ and the orbital period is 7.75 hours.
- Although the orbit is elliptical,

$$\begin{aligned}\frac{dP}{dt} &= -\frac{96\pi}{5}4^{1/3}\left(\frac{2\pi M}{P}\right)^{5/3} \\ &= -3.4\times 10^{-12}\left(\frac{M}{M_\odot}\right)^{5/3}\left(\frac{1\ \mathrm{h}}{P}\right)^{5/3}\end{aligned}$$

  (which we derived for circular orbits) is approximately valid.

- This may be used to estimate that

$$\frac{dP}{dt} \sim -2\times 10^{-13}.$$

- Thus, over a period of one year

$$\Delta P \simeq -6.2\times 10^{-6}\ \mathrm{s}.$$

This shift is tiny but easily measured in the Binary Pulsar system because the timing afforded by the pulsar clock allows orbits to be determined very precisely.

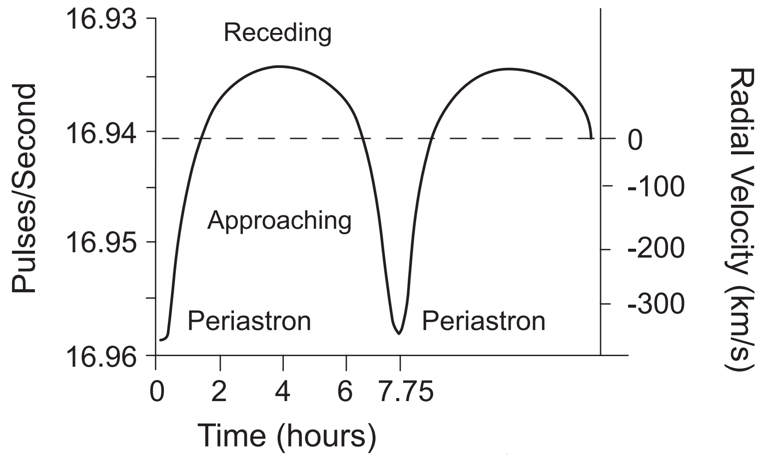


Figure 21.10: Pulse rate and inferred radial velocity as a function of time for the Binary Pulsar.

Figure 21.10 illustrates the pulse rate and the radial velocity as a function of time for the Binary Pulsar.

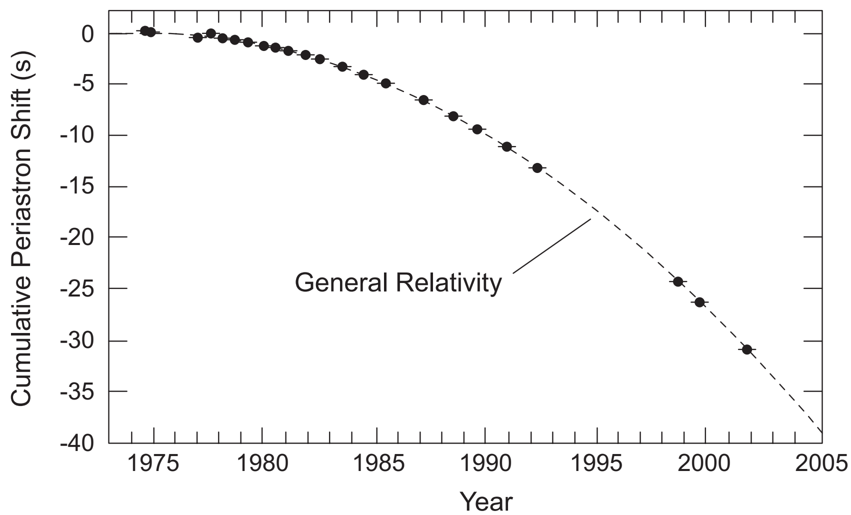


Figure 21.11: Periastron shift of the Binary Pulsar orbit because of gravitational wave emission. Data points are the measurements of Taylor and Weisberg and the dashed curve is the prediction from general relativity.

Fig. 21.11 illustrates the cumulative shift of the periastron time (time for closest approach of the pulsar to its companion) as a function of elapsed time, compared with a precise calculation assuming emission of gravitational waves from the system to be responsible for this shift.

- The quality of the data (note the error bars), and the agreement with the prediction of general relativity are remarkable.
- The Binary Pulsar provides the strongest evidence that gravitational waves exist with properties in quantitative agreement with the predictions of general relativity.

We have not yet seen gravitational waves directly but the Binary Pulsar leaves little doubt of their existence.

21.7 Gravitational Waves from Strong Sources

Our discussion to this point has been primarily in terms of linearized gravity

- However, the sources that are conjectured to provide the most likely events that LIGO and comparable detectors could see cannot be described in the near-source region by a linear approximation to gravity.
- In events such as
  - the core collapse of a massive star, or
  - mergers involving some combination of neutron stars and black holes,

  the curvature of spacetime becomes very large in the region where gravitational waves are produced.
- In this strong-gravity domain, reliable calculations are difficult and can only be produced through large-scale numerical simulations.
- Nevertheless, those simulations show that many of the features that we have inferred about gravity waves in the linear regime survive in some form in the strong-gravity regime.
- In particular, available calculations suggest that many basic features of gravity wave production by strong gravity can be obtained by dimensional analysis.

In this section we summarize some likely strong sources of gravitational waves and show a few results of numerical simulations.

21.7.1 Merging Neutron Stars

Gravitational waves from merging neutron stars are expected to be strong enough that their characteristic signature will be detectable in new Earth-based gravitational wave detectors that are just beginning to operate.

- The possibility of two neutron stars merging might seem a remote one.
- A critical point is that once a neutron star binary is formed its orbital motion radiates energy as gravitational waves, the orbits must shrink, and eventually the two neutron stars must merge.
- Formation of the neutron star binary is not easy, however.
  - A binary must form with two stars massive enough to become supernovae and produce neutron stars, and the neutron stars thus formed must remain bound to each other through the two supernova explosions, or
  - the neutron star binary must result from gravitational capture.
- Although these are improbable scenarios, calculations indicate that they are not impossible.
- Some theoretical estimates indicate that formation of a neutron star binary can happen often enough to produce about one neutron star merger each day in the observable Universe.

The probability of forming a neutron star binary in any one region of space is very small, but the Universe is a very big place.

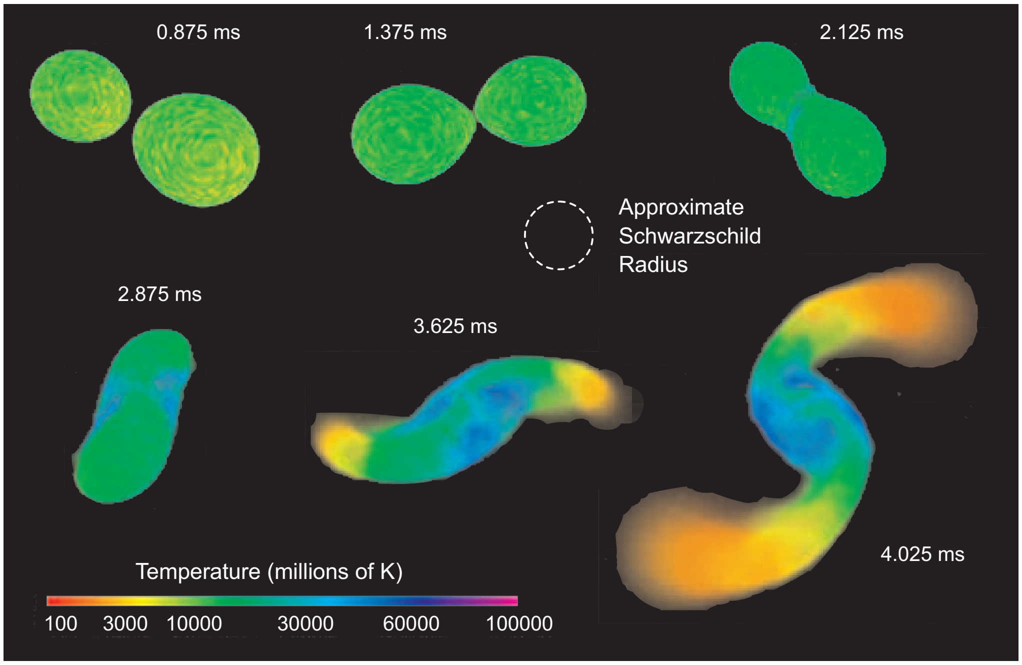
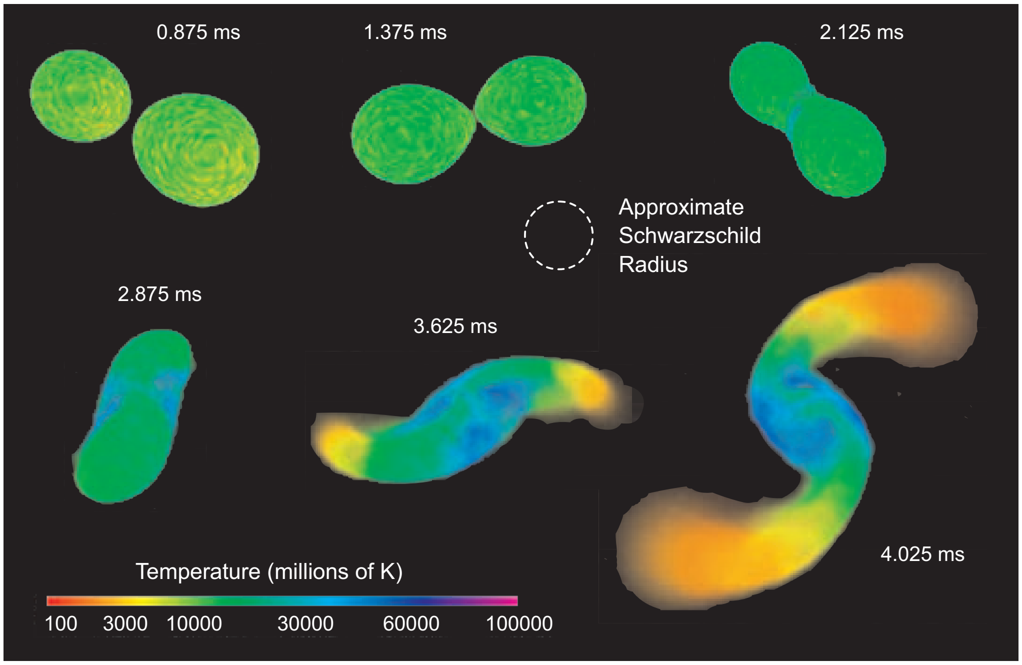


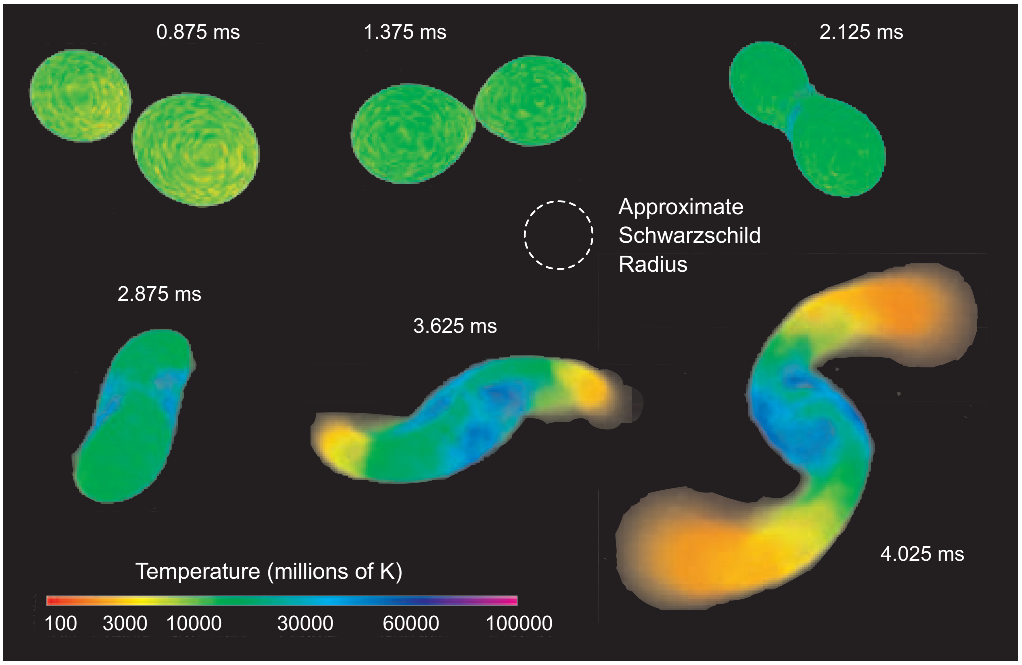
Figure 21.12: Simulation of the merger of two neutron stars. The elapsed time is about 3 ms and the approximate Schwarzschild radius for the combined system is indicated. The rapid motion of several solar masses of material with large quadrupole distortion and sufficient density to be compressed near the Schwarzschild radius indicates that this merger should be a strong source of gravitational waves (Source: S. Rosswog simulation).

Figure 21.12 shows a numerical simulation of neutron-star merger in a binary neutron star system.



In this merger,

- The orbit of the binary has steadily decayed as a result of gravitational wave emission, causing the stars to spiral together at a rapidly increasing rate near the end.
- The sequence of images in Fig. 21.12 illustrates the merger over a period of milliseconds near when the surfaces of the neutron stars first touch.



Because of the

- very large quadrupole mass distortion,
- the high velocities generated by the revolution on millisecond timescales, and
- the highly compact nature of the mass distribution,

mergers of neutron stars are expected to be a very strong source of gravitational waves.

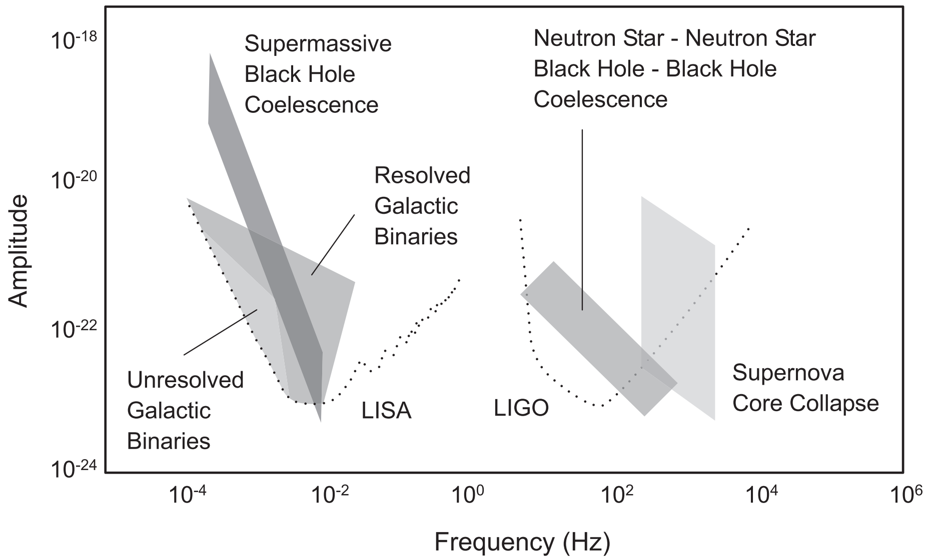


Figure 21.13: Amplitude and frequency ranges expected for gravitational waves from various sources. The lower detection ranges for LIGO and LISA are indicated by dotted lines.

As Fig. 21.13 illustrates,

- the frequencies for gravitational waves emitted from neutron star mergers (and from the stellar-size black holes mergers described below) are expected to lie in the range accessible to Earth-based interferometers like LIGO.
- The outcome of such neutron star mergers will likely be a Kerr black hole, since
  - the combined mass of the two neutron stars is generally larger than the upper limit for a neutron star and
  - the merged object will be formed with large angular momentum.

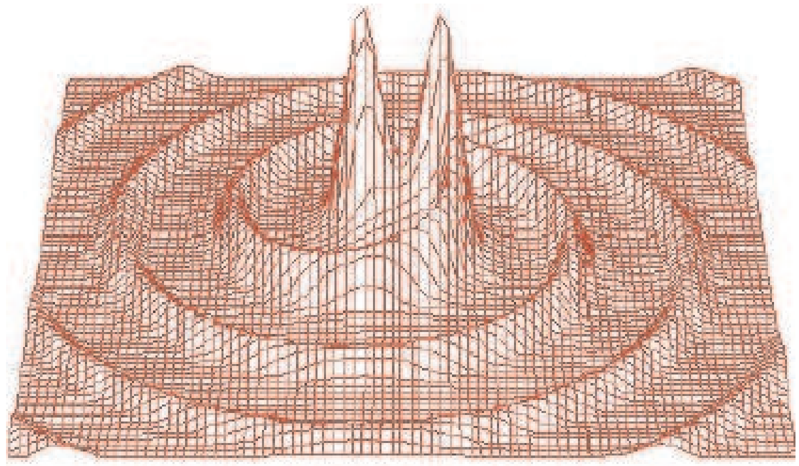


Figure 21.14: Gravitational wave emission from an in-spiraling binary black holes.

21.7.2 Stellar-Size Black Hole Mergers

Merger of 2 black holes, or a neutron star and a black hole, should also be strong sources of gravitational waves.

- Such mergers are expected to have many qualitative similarities to the merger of neutron stars.
- They are expected to leave behind a Kerr black hole.

Figure 21.14 illustrates gravitational wave emission from an inspiraling black hole pair based on numerical simulations of general relativity.

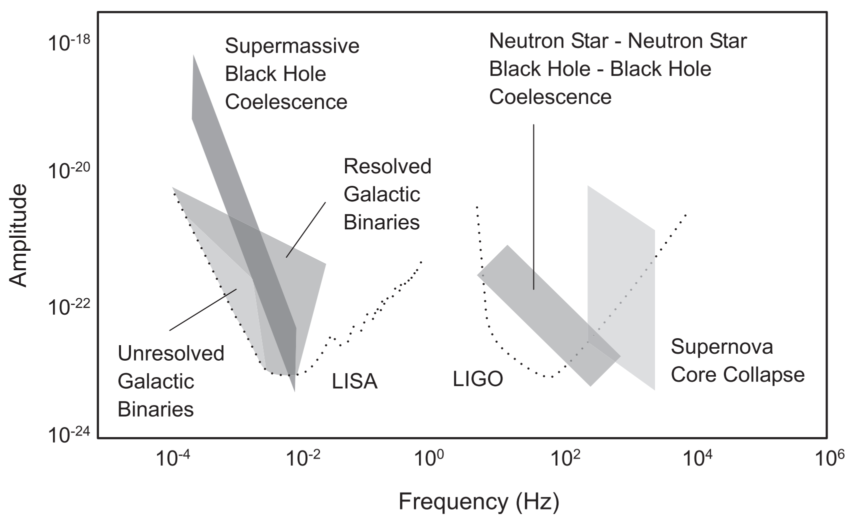
- The two strong peaks represent the black holes.
- The spirals in spacetime trailing from them are the outwardly propagating gravitational waves.

21.7.3 Core Collapse in Massive Stars

In the core collapse of a massive star that leads to a supernova

- Densities are reached comparable to that of a neutron star and
- in the collapse a solar mass or more of material may be set in motion with velocities of order 10% of light velocity.
- If such a collapse were to proceed with spherical symmetry, no gravitational waves would be produced.
- However, numerical simulations generally find large asymmetries generated by things like large-scale supersonic convection.
- These asymmetries will likely produce quadrupole distortions in the mass distribution that vary rapidly in time and thus are potential sources of strong gravitational waves.

As indicated in the following figure,



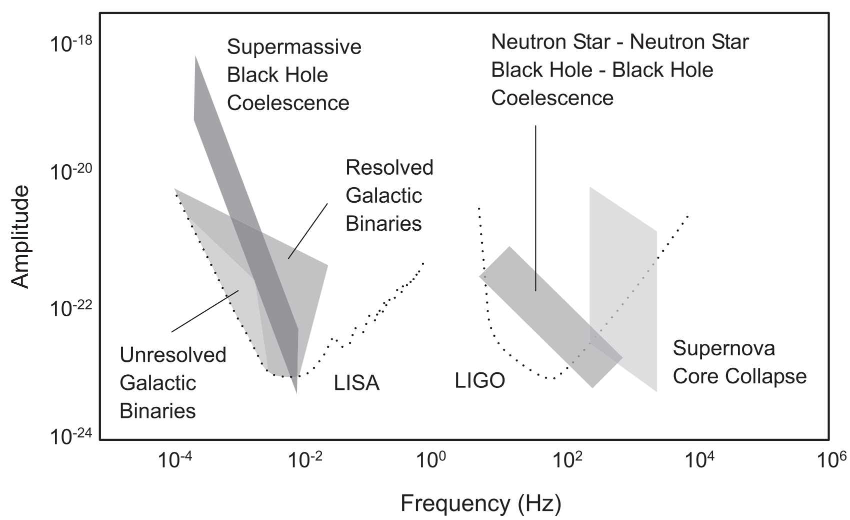
gravitational-wave frequencies from core-collapse supernovae are expected to lie in the range accessible to Earth-based interferometers.

21.7.4 Merging Supermassive Black Holes

The current paradigm is that the cores of many, perhaps most, massive galaxies harbor black holes containing 10^6–10^9 solar masses.

- For example, fully-resolved orbits of individual stars at the center of the Milky Way indicate a $3.3 \times 10^6 M_\odot$ black hole.
- Analysis of velocity fields near the centers of giant elliptical galaxies often suggest billions of solar masses packed into non-luminous regions comparable in size to the Solar System.
- There is also very strong observational evidence that galaxy mergers are common in the history of the Universe.
- Therefore, we may expect that the merger of supermassive black holes can occur as a consequence of galaxy collisions.

As indicated in the following figure,



The expected frequency of the gravitational waves that would be produced is too low for detectors like LIGO but the proposed space-based LISA array would be optimal to search for such events.

Quantitative investigation of supermassive black hole merger events can only be done numerically but dimensional analysis arguments based on the quadrupole power formula

$$L = \frac{dE}{dt} = \frac{1}{5}\frac{G}{c^5}\left\langle \dddot{I\!\!\!-}_{ij}\dddot{I\!\!\!-}^{ij} \right\rangle,$$

and supported by numerical simulation suggest that the peak luminosity is set by the fundamental scale c^5/G defined in

$$L \simeq L_0 \frac{r_S^2}{R^2}\left(\frac{v}{c}\right)^6 \qquad L_0 \equiv \frac{c^5}{G} = 3.6 \times 10^{59} \text{ erg s}^{-1}$$

Current understanding indicates that the peak luminosity for supermassive black hole mergers is approximately

$$L \propto L_0 \simeq \eta \frac{c^5}{G} \simeq \eta \times 10^{59} \text{ erg s}^{-1},$$

with the efficiency factor η accounting for details of the merger and being of order 1% in typical cases.

> If such events occur, their luminosities would likely be *larger than for any other event in the Universe* for a period of days.

To set the gravitational-wave luminosity

$$L \propto L_0 \simeq \eta \frac{c^5}{G} \simeq \eta \times 10^{59} \text{ erg s}^{-1},$$

expected for merger of supermassive black holes in perspective, this luminosity (assuming $\eta \sim 0.01$) is about

- a million times larger than the photon luminosities associated with supernovae and gamma ray bursts,
- ten billion times larger than quasar luminosities, and
- about 25 orders of magnitude larger than the luminosity of the Sun.