Chapter 4

Lecture: Lorentz Covariance

To go beyond Newtonian gravitation we must consider, with Einstein, the intimate relationship between the curvature of space and the gravitational field.

- Mathematically, this extension is bound inextricably to the *geometry of spacetime*, and in particular to the aspect of geometry that permits quantitative measurement of distances.
- Let us first consider these ideas within the 4-dimensional spacetime termed *Minkowski space*.

As we shall see, requiring covariance within Minkowski space will lead us to the *special theory of relativity*.

4.0.4 Minkowski Space

In a particular inertial frame, introduce unit vectors e_0, e_1, e_2, and e_3 that point along the t, x, y, and z axes. Any 4-vector b may be expressed in the form,

$$b = b^0 e_0 + b^1 e_1 + b^2 e_2 + b^3 e_3.$$

and the scalar product of 4-vectors is given by

$$a \cdot b = b \cdot a = (a^\mu e_\mu) \cdot (b^\nu e_\nu) = e_\mu \cdot e_\nu a^\mu b^\nu.$$

Note that generally we shall use

- non-bold symbols to denote 4-vectors
- bold symbols for 3-vectors.

Where there is potential for confusion, we use a notation such as b^μ to stand generically for all components of a 4-vector.

Introducing the definition

$$\eta_{\mu\nu} \equiv e_\mu \cdot e_\nu,$$

the scalar product may be expressed as

$$a \cdot b = \eta_{\mu\nu} a^\mu b^\nu.$$

and thus the line element becomes

$$ds^2 = -c^2dt^2 + dx^2 + dy^2 + dz^2 = \eta_{\mu\nu}dx^\mu dx^\nu,$$

where the *metric tensor of flat spacetime* may be expressed as

$$\eta_{\mu\nu} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \mathrm{diag}\,(-1,1,1,1).$$

That is, the line element corresponds to the matrix equation

$$ds^2 = (cdt \;\; dx \;\; dy \;\; dz) \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} cdt \\ dx \\ dy \\ dz \end{pmatrix},$$

where ds^2 represents the spacetime interval between x and $x + dx$ with

$$x = (x^0, x^1, x^2, x^3) = (ct, x^1, x^2, x^3).$$

The Minkowski metric is sometimes termed a *pseudo-euclidean metric* to emphasize that it is euclidean-like except for the difference in sign between the time and space terms in the line element.

Example 4.1

Let us use the metric to determine the relationship between the time coordinate t and the proper time τ, with $\tau^2 \equiv -s^2/c^2$. From

$$ds^2 = -c^2 dt^2 + dx^2 + dy^2 + dz^2,$$

we may write

$$\begin{aligned} d\tau^2 &= \frac{-ds^2}{c^2} = \frac{1}{c^2}(c^2 dt^2 - dx^2 - dy^2 - dz^2) \\ &= dt^2 \left\{ 1 - \frac{1}{c^2} \underbrace{\left[\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2 \right]}_{v^2} \right\} \\ &= \left(1 - \frac{v^2}{c^2}\right) dt^2. \end{aligned}$$

where v is the magnitude of the velocity. Therefore, the *proper time* τ that elapses between *coordinate times* t_1 and t_2 is

$$\tau_{12} = \int_{t_1}^{t_2} \left(1 - \frac{v^2}{c^2}\right)^{1/2} dt.$$

The proper time interval τ_{12} is shorter than the coordinate time interval $t_2 - t_1$ because the square root is always less than one. This is the time dilation effect of special relativity, stated in general form. If the velocity is constant, this reduces to

$$\Delta\tau = \left(1 - \frac{v^2}{c^2}\right)^{1/2} \Delta t,$$

which is the usual statement of *time dilation in special relativity*.

Table 4.1: Rank 0, 1, and 2 tensor transformation laws

Tensor	Transformation law
Scalar	$\varphi' = \varphi$
Covariant vector	$A'_\mu = \dfrac{\partial x^\nu}{\partial x'^\mu} A_\nu$
Contravariant vector	$A'^\mu = \dfrac{\partial x'^\mu}{\partial x^\nu} A^\nu$
Covariant rank-2	$T'_{\mu\nu} = \dfrac{\partial x^\alpha}{\partial x'^\mu} \dfrac{\partial x^\beta}{\partial x'^\nu} T_{\alpha\beta}$
Contravariant rank-2	$T'^{\mu\nu} = \dfrac{\partial x'^\mu}{\partial x^\alpha} \dfrac{\partial x'^\nu}{\partial x^\beta} T^{\alpha\beta}$
Mixed rank-2	$T'^{\nu}{}_{\mu} = \dfrac{\partial x^\alpha}{\partial x'^\mu} \dfrac{\partial x'^\nu}{\partial x^\beta} T^{\beta}{}_{\alpha}$

4.1 Tensors in Minkowski Space

In Minkowski space, *transformations between coordinate systems are independent of spacetime.* Thus derivatives appearing in the general definitions of Table 4.1 for tensors are constants and for flat spacetime we have the simplified tensor transformation laws

$$
\begin{aligned}
\varphi' &= \varphi && \text{Scalar} \\
A'^\mu &= \Lambda^\mu{}_\nu A^\nu && \text{Contravariant vector} \\
A'_\mu &= \Lambda_\mu{}^\nu A_\nu && \text{Covariant vector} \\
T'^{\mu\nu} &= \Lambda^\mu{}_\gamma \Lambda_\delta{}^\nu T^{\gamma\delta} && \text{Contravariant rank-2 tensor} \\
T'_{\mu\nu} &= \Lambda_\mu{}^\gamma \Lambda^\delta{}_\nu T_{\gamma\delta} && \text{Covariant rank-2 tensor} \\
T'^\mu{}_\nu &= \Lambda^\mu{}_\gamma \Lambda^\delta{}_\nu T^\gamma{}_\delta && \text{Mixed rank-2 tensor}
\end{aligned}
$$

where the matrix $\Lambda^\mu{}_\nu$ does not depend on the spacetime coordinates.

In addition, for flat spacetime we may use a coordinate system for which the second term of

$$A'_{\mu,\nu} = \underbrace{A_{\alpha,\beta} \frac{\partial x^\beta}{\partial x'^\nu} \frac{\partial x^\alpha}{\partial x'^\mu}}_{\textit{Tensor}} + \underbrace{A_\alpha \frac{\partial^2 x^\alpha}{\partial x'^\nu \partial x'^\mu}}_{\textit{Not a tensor}}$$

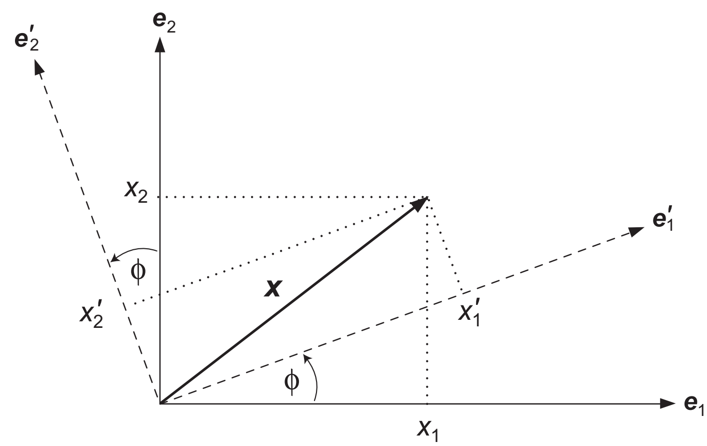
can be transformed away and in flat spacetime *covariant derivatives are equivalent to partial derivatives.*

In the Minkowski transformation laws the $\Lambda^\mu{}_\nu$ are elements of *Lorentz transformations,* to which we now turn our attention.

4.2 Lorentz Transformations

In 3-dimensional euclidean space, rotations are a particularly important class of transformations because they change the direction for a 3-vector but preserve its length.

- We wish to generalize this idea to investigate abstract rotations in the 4-dimensional Minkowski space.
- Such rotations in Minkowski space are termed *Lorentz transformations.*



Consider a rotation of the coordinate system in euclidean space, as illustrated in the figure above.

- For the length of an arbitrary vector $\boldsymbol{x}$ to be unchanged by this transformation means that $\boldsymbol{x}\cdot\boldsymbol{x} = \boldsymbol{x}'\cdot\boldsymbol{x}'$, which (since $\boldsymbol{x}\cdot\boldsymbol{x} = g_{ij}x^i x^j$) requires that the transformation matrix R implementing the rotation $x'^i = R^i_j x^j$ act on the metric tensor g_{ij} in the following way

$$Rg_{ij}R^{\mathrm{T}} = g_{ij},$$

  where R^{T} denotes the transpose of R.

- For euclidean space the metric tensor is just the unit matrix so the above requirement reduces to $RR^{\mathrm{T}} = 1$, which is the condition that R be an *orthogonal matrix*.

- Thus, we obtain by this somewhat pedantic route the well-known result that *rotations in euclidean space are implemented by orthogonal matrices*.

- But the requirement $Rg_{ij}R^{\mathrm{T}} = g_{ij}$ for rotations is *valid generally*, not just for euclidean spaces. Therefore, let us use it as guidance to constructing *generalized rotations in Minkowski space*.

- By analogy with the above discussion of rotations in euclidean space, we seek a set of transformations that leave the length of a 4-vector invariant in the Minkowski space.

- We write the coordinate transformation in matrix form,

$$dx'^{\mu} = \Lambda^{\mu}{}_{\nu} dx^{\nu},$$

  where we expect the transformation matrix $\Lambda^{\mu}{}_{\nu}$ to satisfy the analog of $Rg_{ij}R^{\mathrm{T}} = g_{ij}$ for the Minkowski metric $\eta_{\mu\nu}$,

$$\Lambda \eta_{\mu\nu} \Lambda^{\mathrm{T}} = \eta_{\mu\nu},$$

  Or explicitly in terms of components, $\Lambda_{\mu}{}^{\rho} \Lambda^{\sigma}{}_{\nu} \eta_{\rho\sigma} = \eta_{\mu\nu}$.

- Let us now use this property to construct the elements of the transformation matrix $\Lambda^{\mu}{}_{\nu}$ These will include

  – rotations about the spatial axes (corresponding to rotations within inertial systems) and

  – transformations between inertial systems moving at different constant velocities that are termed *Lorentz boosts*.

  We consider first the simple case of rotations about the z axis.

4.2.1 Rotations

For rotations about the z axis The transformation may we written in matrix notation as

$$\begin{pmatrix} x'^1 \\ x'^2 \end{pmatrix} = R \begin{pmatrix} x^1 \\ x^2 \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x^1 \\ x^2 \end{pmatrix}.$$

where a, b, c, and d parameterize the transformation matrix.

- Rotations about a single axis correspond to a 2-dimensional problem with euclidean metric, so the condition $Rg_{ij}R^{\mathrm{T}} = g_{ij}$ is

$$\underbrace{\begin{pmatrix} a & b \\ c & d \end{pmatrix}}_{R} \underbrace{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}}_{g_{ij}} \underbrace{\begin{pmatrix} a & c \\ b & d \end{pmatrix}}_{R^{\mathrm{T}}} = \underbrace{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}}_{g_{ij}},$$

- Carrying out the matrix multiplications on the left side gives

$$\begin{pmatrix} a^2+b^2 & ac+bd \\ ac+bd & c^2+d^2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

and comparison of the two sides of the equation implies that

$$a^2+b^2 = 1 \qquad c^2+d^2 = 1 \qquad ac+bd = 0.$$

- These requirements are satisfied by the choices

$$a = \cos\varphi \qquad b = \sin\varphi \qquad c = -\sin\varphi \qquad d = \cos\varphi,$$

and we obtain the expected result for an ordinary rotation,

$$\begin{pmatrix} x'^1 \\ x'^2 \end{pmatrix} = R \begin{pmatrix} x^1 \\ x^2 \end{pmatrix} = \begin{pmatrix} \cos\varphi & \sin\varphi \\ -\sin\varphi & \cos\varphi \end{pmatrix} \begin{pmatrix} x^1 \\ x^2 \end{pmatrix}.$$

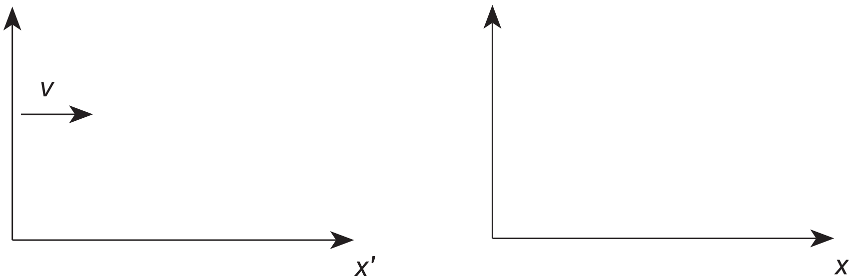


Figure 4.1: A Lorentz boost along the positive x axis.

Now, let's apply this same technique to determine the elements of a *Lorentz boost transformation*.

4.2.2 Lorentz Boosts

Consider a boost from one inertial system to a 2nd one moving in the positive direction at uniform velocity along the x axis (Fig. 4.1).

- The y and z coordinates are unaffected by this boost, so this also is effectively a 2-dimensional transformation on t and x,

$$\begin{pmatrix} cdt' \\ dx' \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} cdt \\ dx \end{pmatrix}$$

- We can write the condition $\Lambda \eta_{\mu\nu} \Lambda^{\mathrm{T}} = \eta_{\mu\nu}$ out explicitly as

$$\underbrace{\begin{pmatrix} a & b \\ c & d \end{pmatrix}}_{\Lambda} \underbrace{\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}}_{\eta_{\mu\nu}} \underbrace{\begin{pmatrix} a & c \\ b & d \end{pmatrix}}_{\Lambda^{\mathrm{T}}} = \underbrace{\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}}_{\eta_{\mu\nu}},$$

(identical to the rotation case, except for the *indefinite metric*).

- Multiplying the matrices on the left side and comparing with the matrix on the right side in

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a & c \\ b & d \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix},$$

gives the conditions

$$a^2 - b^2 = 1 \qquad -c^2 + d^2 = 1 \qquad -ac + bd = 0,$$

- These are satisfied if we choose

$$a = \cosh\xi \qquad b = \sinh\xi \qquad c = \sinh\xi \qquad d = \cosh\xi,$$

where ξ is a hyperbolic variable taking the values $-\infty \leq \xi \leq \infty$.

- Therefore, the boost transformation may be written as

$$\begin{pmatrix} cdt' \\ dx' \end{pmatrix} = \begin{pmatrix} \cosh\xi & \sinh\xi \\ \sinh\xi & \cosh\xi \end{pmatrix} \begin{pmatrix} cdt \\ dx \end{pmatrix}.$$

Which may be compared with the rotational result

$$\begin{pmatrix} x'^1 \\ x'^2 \end{pmatrix} = \begin{pmatrix} \cos\varphi & \sin\varphi \\ -\sin\varphi & \cos\varphi \end{pmatrix} \begin{pmatrix} x^1 \\ x^2 \end{pmatrix}.$$

Box 4.1 Minkowski Rotations

The respective derivations make clear that the appearance of hyperbolic functions in the boosts, rather than trigonometric functions as in rotations, traces to the role of the indefinite metric $g_{\mu\nu} = \text{diag}\,(-1,1)$ in the boosts.

- The hyperbolic functions suggest that the boost transformations are "rotations"in Minkowski space.
- But these rotations
  - mix space and time, and
  - will have unusual properties since they correspond to rotations through imaginary angles (see Exercise 4.6).
- These unusual properties follow from the metric:
  - The invariant interval that is being conserved is not the length of vectors in space or the length of time intervals separately.
  - Rather it is the specific *mixture of time and space intervals* implied by the Minkowski line element with *indefinite metric*:

$$ds^2 = (cdt \;\; dx \;\; dy \;\; dz) \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} cdt \\ dx \\ dy \\ dz \end{pmatrix},$$

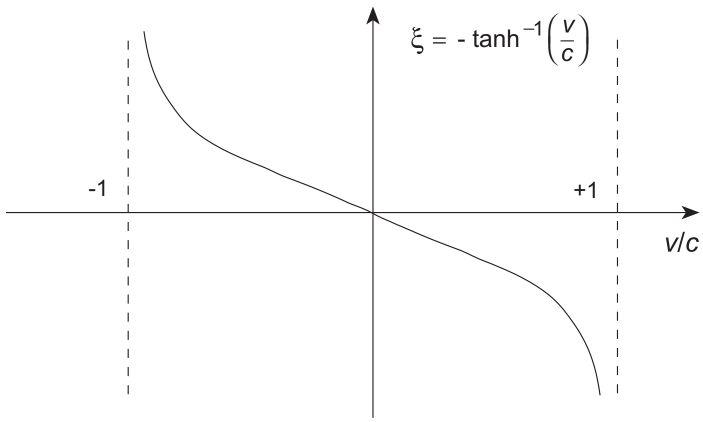


Figure 4.2: Dependence of the Lorentz parameter ξ on $\beta = v/c$.

We can put the Lorentz boost transformation into a more familiar form by relating the boost parameter ξ to the boost velocity.

- The velocity of the boosted system is $v = x/t$. From

$$\begin{pmatrix} cdt' \\ dx' \end{pmatrix} = \begin{pmatrix} \cosh\xi & \sinh\xi \\ \sinh\xi & \cosh\xi \end{pmatrix} \begin{pmatrix} cdt \\ dx \end{pmatrix},$$

  the origin ($x' = 0$) of the boosted system is

$$x' = ct\sinh\xi + x\cosh\xi = 0.$$

- Therefore, $x/t = -c\sinh\xi/\cosh\xi$, from which we conclude that

$$\beta \equiv \frac{v}{c} = \frac{x}{ct} = -\frac{\sinh\xi}{\cosh\xi} = -\tanh\xi.$$

- This relationship between ξ and β is plotted in Fig. 4.2.

- Utilizing
$$\beta \equiv \frac{v}{c} = \frac{x}{ct} = -\frac{\sinh\xi}{\cosh\xi} = -\tanh\xi.$$
the identity $\cosh^2\xi - \sinh^2\xi = 1$, and the definition
$$\gamma \equiv \left(1 - \frac{v^2}{c^2}\right)^{-1/2}$$
of the Lorentz γ factor, we may write
$$\cosh\xi = \frac{1}{\sqrt{1-\sinh^2\xi/\cosh^2\xi}} = \frac{1}{\sqrt{1-v^2/c^2}} = \gamma,$$

- From this result and
$$\beta = -\frac{\sinh\xi}{\cosh\xi}$$
we obtain
$$\sinh\xi = -\beta\cosh\xi = -\beta\gamma.$$

- Thus, inserting $\cosh\xi = \gamma$ and $\sinh\xi = -\beta\gamma$ in the Lorentz transformation gives
$$\begin{pmatrix} cdt' \\ dx' \end{pmatrix} = \begin{pmatrix} \cosh\xi & \sinh\xi \\ \sinh\xi & \cosh\xi \end{pmatrix}\begin{pmatrix} cdt \\ dx \end{pmatrix}$$
$$= \begin{pmatrix} \gamma & -\gamma\beta \\ -\gamma\beta & \gamma \end{pmatrix}\begin{pmatrix} cdt \\ dx \end{pmatrix}.$$
$$= \gamma\begin{pmatrix} 1 & -\beta \\ -\beta & 1 \end{pmatrix}\begin{pmatrix} cdt \\ dx \end{pmatrix}.$$

- Writing the matrix expression

$$\begin{pmatrix} cdt' \\ dx' \end{pmatrix} = \gamma \begin{pmatrix} 1 & -\beta \\ -\beta & 1 \end{pmatrix} \begin{pmatrix} cdt \\ dx \end{pmatrix}.$$

  out explicitly for finite intervals gives the Lorentz boost equations (for the specific case of a positive boost along the x axis) in standard textbook form,

$$t' = \gamma\left(t - \frac{vx}{c^2}\right)$$
$$x' = \gamma(x - vt)$$
$$y' = y \qquad z' = z,$$

- The inverse transformation corresponds to the replacement $v \to -v$.

- Clearly these reduce to the Galilean boost equations

$$\boldsymbol{x}' = \boldsymbol{x}'(\boldsymbol{x}, t) = \boldsymbol{x} - \boldsymbol{v}t \qquad t' = t'(\boldsymbol{x}, t) = t.$$

  in the limit that v/c vanishes, as we would expect.

- It is easily verified (Exercise 4.1) that the Lorentz transformations leave invariant the spacetime interval ds^2.

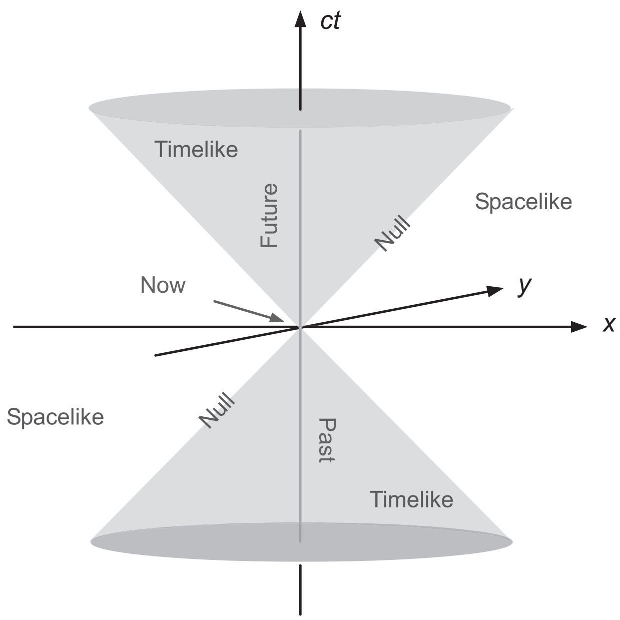


Figure 4.3: The light cone diagram for two space and one time dimensions.

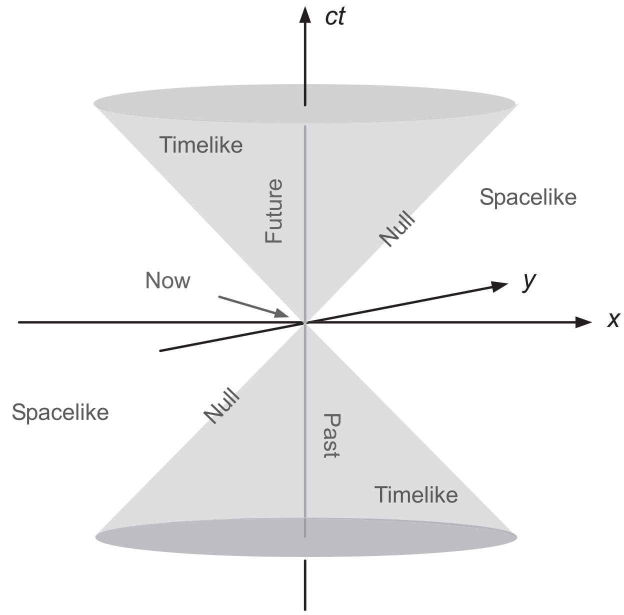
4.2.3 The Light Cone Structure of Minkowski Spacetime

By virtue of the line element (which defines a cone)

$$ds^2 = -c^2 dt^2 + dx^2 + dy^2 + dz^2,$$

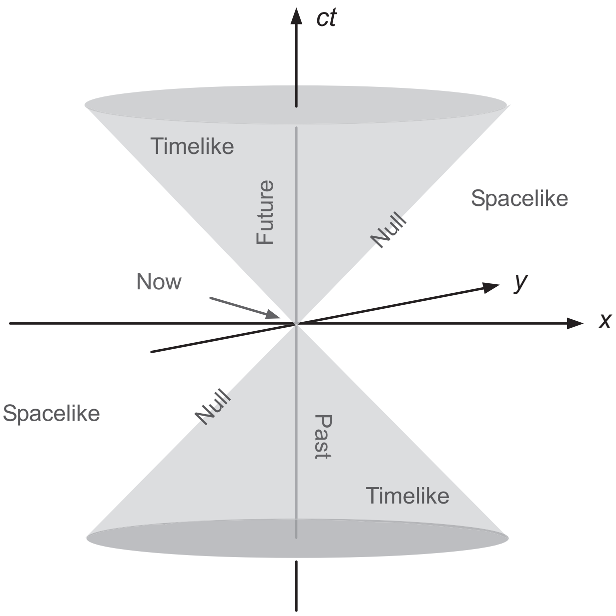
the Minkowski spacetime may be classified according to the light cone diagram exhibited in Fig. 4.3.

The light cone is a 3-dimensional surface in the 4-dimensional spacetime and events in spacetime may be characterized according to whether they are inside of, outside of, or on the light cone.



The standard terminology (assuming our $(-1,1,1,1)$ metric signature):

- If $ds^2 < 0$ the interval is termed *timelike*. In that case ds/c is the time measured by a clock moving freely from x to $x+dx$ (the proper time; see below).
- If $ds^2 > 0$ the interval is termed *spacelike*. Then $|ds^2|^{1/2}$ may be interpreted as the length of a ruler with ends at x and $x+dx$, as measured by an observer at rest with respect to the ruler.
- If $ds^2 = 0$ the interval is called *lightlike* (or sometimes *null*). Then the points x and $x+dx$ are connected by signals moving at light speed.



The light cone classification clarifies the distinction between Minkowski spacetime and a 4D euclidean space in that two points in the Minkowski spacetime may be separated by a distance whose square could be

- positive,
- negative, or
- zero

which embodies impossibilities for a euclidean space.

In particular, lightlike particles have worldlines confined to the light cone and the square of the separation of any two points on a lightlike worldline is *zero*.

Example 4.2

The Minkowski line element in one space and one time dimension [often termed $(1+1)$ dimensions] is $ds^2 = -c^2dt^2 + dx^2$. Thus, if $ds^2 = 0$

$$-c^2dt^2 + dx^2 = 0 \quad \longrightarrow \quad \left(\frac{dx}{dt}\right)^2 = c^2 \quad \longrightarrow \quad v = \pm c.$$

We can generalize this result easily to the full space and we conclude that

- Events in Minkowski space separated by a null interval ($ds^2 = 0$) are connected by signals moving at light velocity, $v = c$.
- If the time (ct) and space axes have the same scales, this means that the worldline of a freely propagating photon (or any massless particle moving at light velocity) always make $\pm 45^\circ$ angles in the lightcone diagram.
- By similar arguments, events at timelike separations (inside the lightcone) are connected by signals with $v < c$, and
- Those with spacelike separations (outside the lightcone) could be connected only by signals with $v > c$ (which would violate causality).

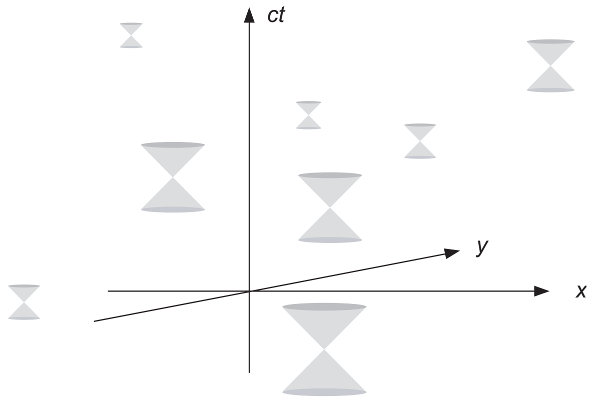


Figure 4.4: We may imagine a lightcone attached to every point of spacetime.

We have placed the lightcone in the earlier illustration at the origin of our coordinate system, but in general we may imagine a lightcone attached to every point in the spacetime, as illustrated in Fig. 4.4.

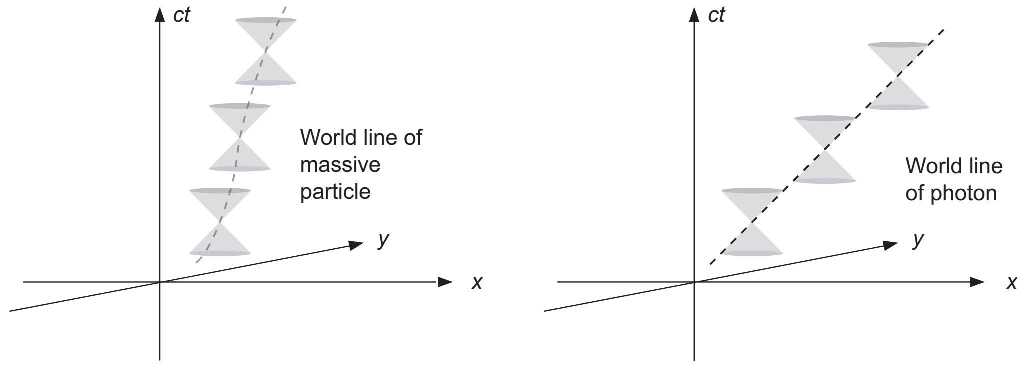


Figure 4.5: Worldlines for massive particles and massless particles like photons.

A tangent to the worldline of any particle defines the local velocity of the particle at that point and constant velocity implies straight worldlines. Therefore, as illustrated in Fig. 4.5,

- Light must always travel in straight lines (in Minkowski space; *not* in curved space), and always on the lightcone, since $v = c =$ constant. Thus photons have constant local velocities.
- Worldlines for any massive particle lie inside the local lightcone since $v \leq c$ (*timelike trajectory*, since always within the lightcone).
- Worldline for the massive particle in this example is curved *(acceleration)*.
- For non-accelerated massive particles the worldline would be straight, but always *within the lightcone.*

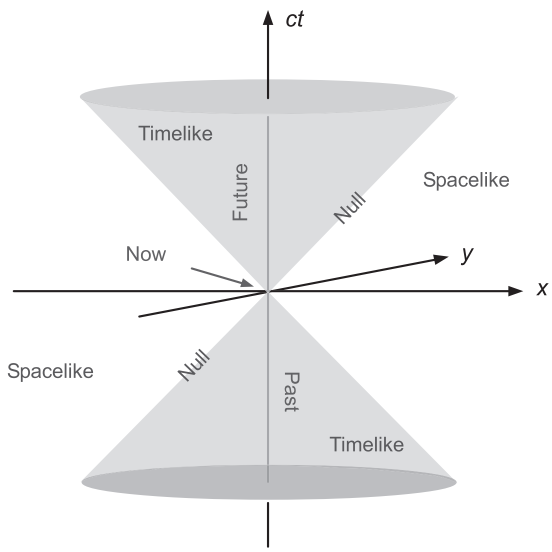


Figure 4.6: The light cone diagram for two space and one time dimensions.

4.2.4 Causality and Spacetime

The causal properties of Minkowski spacetime are encoded in its light cone structure, which requires that $v \leq c$ for all signals.

- Each point in spacetime may be viewed as lying at the apex of a light cone ("Now").
- An event at the origin of a light cone may influence any event in its forward light cone (the "Future").
- The event at the origin of the light cone may be influenced by events in its backward light cone (the "Past").
- Events at spacelike separations are causally disconnected from the event at the origin.
- Events on the light cone are connected by signals that travel exactly at c.

The light cone is a surface separating the knowable from the unknowable for an observer at the apex of the light cone.

This light cone structure of spacetime ensures that all velocities obey locally the constraint $v \leq c$. Since velocities are defined and measured locally, covariant field theories in either flat or curved space are guaranteed to respect the speed limit $v \leq c$, irrespective of whether globally velocities appear to exceed c.

EXAMPLE: In the Hubble expansion of the Universe, galaxies beyond a certain distance (the horizon) would recede from us at velocities in excess of c. However, all local measurements in that expanding, possibly curved, space would determine the velocity of light to be c.

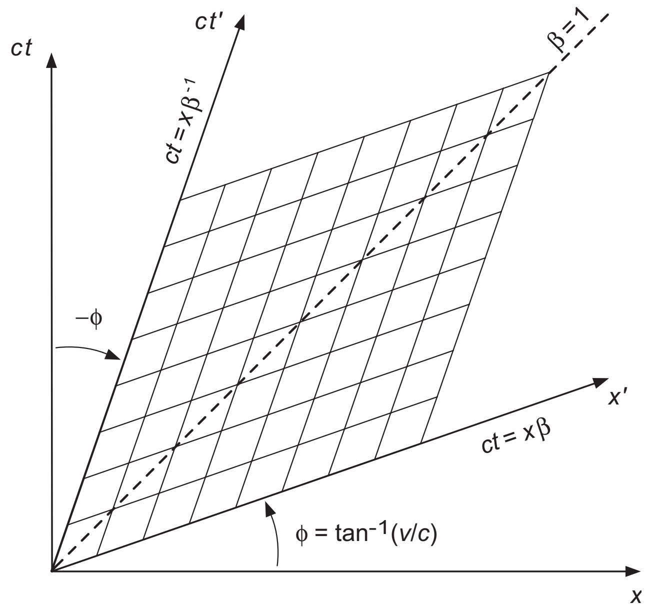
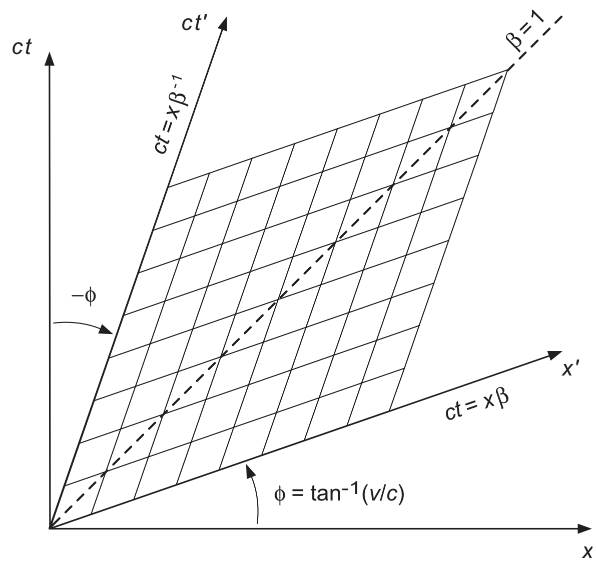


Figure 4.7: Lorentz boost transformation in a spacetime diagram.

4.3 Lorentz Transformations in Spacetime Diagrams

It is instructive to look at the action of Lorentz transformations in the spacetime (lightcone) diagram. If we consider boosts only in the x direction, the relevant part of the spacetime diagram in some inertial frame corresponds to a plot with axes ct and x, as illustrated in the figure above.

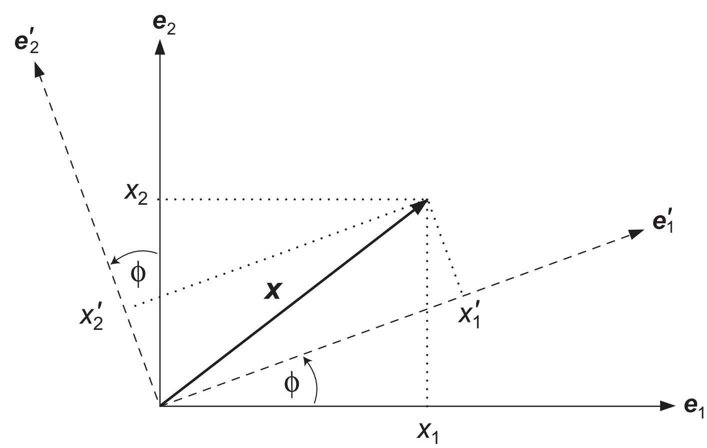


Let us now ask what happens to these axes under the Lorentz boost

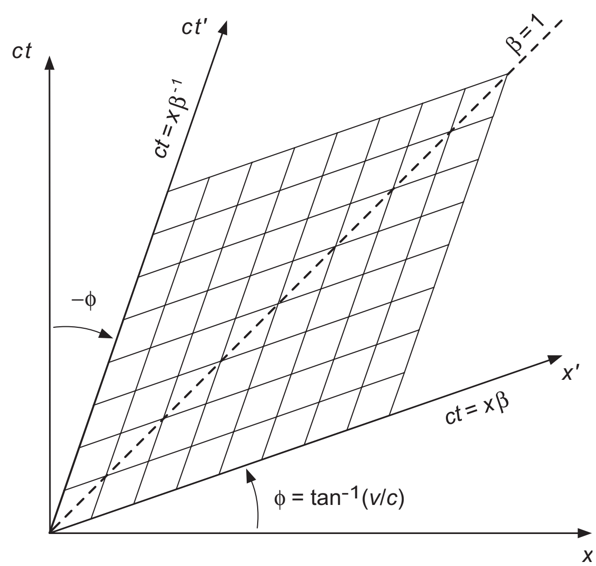
$$ct' = c\gamma\left(t - \frac{vx}{c^2}\right) \qquad x' = \gamma(x - vt).$$

- The t' axis corresponds to $x' = 0$. From the 2nd equation $x = vt \to x/c = (v/c)t = \beta t$ so that $ct = x\beta^{-1}$, where $\beta = v/c$.
- Likewise, the x' axis corresponds to $t' = 0$, which implies from the 1st equation that $ct = (v/c)x = x\beta$.
- Thus, the equations of the x' and t' axes (in the (x, ct) coordinate system) are $ct = x\beta$ and $ct = x\beta^{-1}$, respectively.
- The $x' = 0$ and t' axes for the boosted system are also shown in the figure for a boost corresponding to a positive value of β.
- Time and space axes are rotated by same angle, but in *opposite directions* by the boost (due to the *indefinite Minkowski metric*).
- Rotation angle related to boost velocity through $\tan\varphi = v/c$.

Ordinary rotations (the two axes rotate by the same angle in the *same direction*):



Lorentz boost "rotations" (the two axes rotate by the same angle but in *opposite directions*):



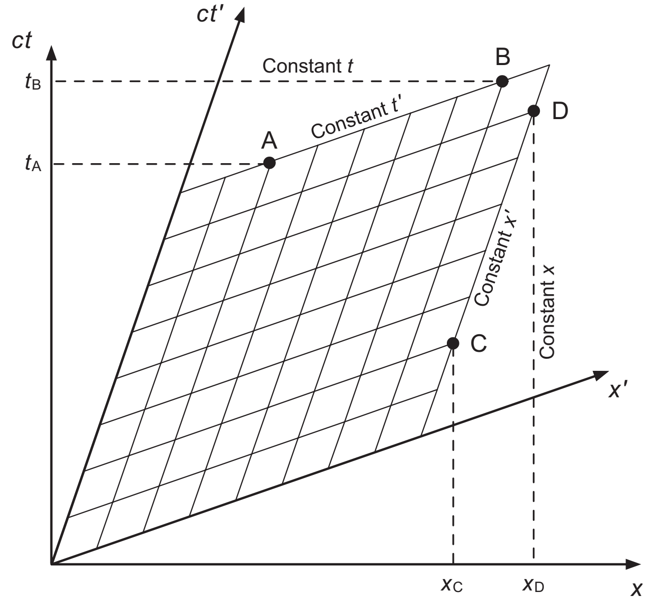


Figure 4.8: Comparison of events in boosted and unboosted reference frames.

Most of special relativity follows directly from this figure.

For example, relativity of simultaneity follows directly, as illustrated in Fig. 4.8.

- Points A and B lie on the same t' line, so they are *simultaneous in the boosted frame*.
- But from the dashed projections on the ct axis, *event A occurs before event B in the unboosted frame*.
- Likewise, points C and D lie at the same value of x' in the boosted frame and so are *spatially congruent*, but in the unboosted frame $x_C \neq x_D$.

Relativistic time dilation and space contraction effects follow rather directly from these observations.

Example 4.3

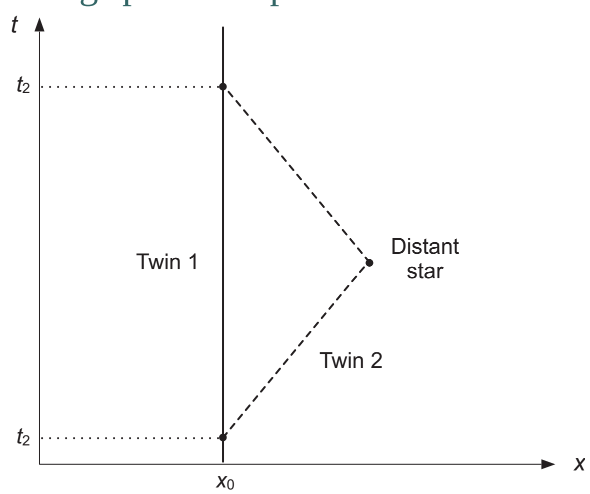
The time registered by a clock moving between two points in spacetime depends on the path followed, as suggested by

$$d\tau^2 = \left(1 - \frac{v^2}{c^2}\right) dt^2.$$

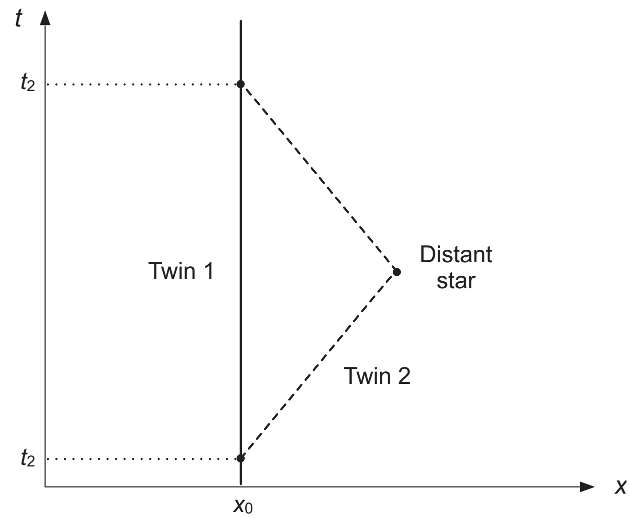
> The proper time τ is the time registered by a clock carried by an observer on a spacetime path.

That this is true even if the path returns to the initial spatial position is the source of the *twin paradox* of special relativity.

- Twins are initially at rest in the same inertial frame. Twin 2 travels at $v \sim c$ to a distant star and then returns at the same speed to the starting point; twin 1 remains at the starting point.
- The corresponding spacetime paths are:



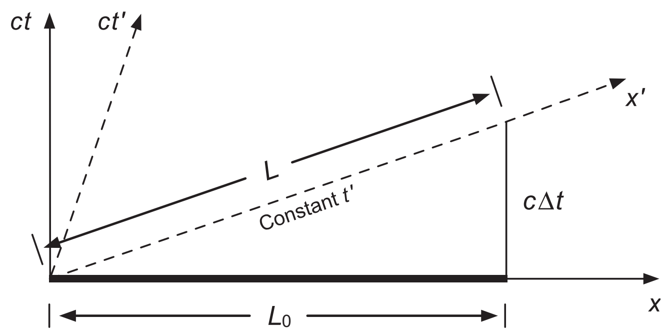
- The elapsed time on the clock carried by Twin 2 is always smaller because of the square root factor in the above equation.



- The (seeming) paradox arises if one describes things from the point of view of Twin 2, who sees Twin 1 move away and then back. This seems to be symmetric with the case of Twin 1 watching Twin 2 move away and then back.
- But it isn't: *the twins travel different worldlines, and different distances along these worldlines.*
- Their clocks record the proper time on their respective worldlines and thus differ when they are rejoined, indicating unambiguously that Twin 2 is younger at the end of the journey.

Space Contraction

Consider the following schematic spacetime diagram, where a rod of *proper length* L_0, as measured in its own rest frame (t, x), is oriented along the x axis.

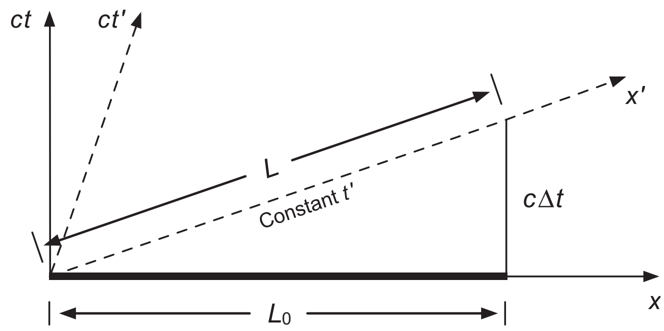


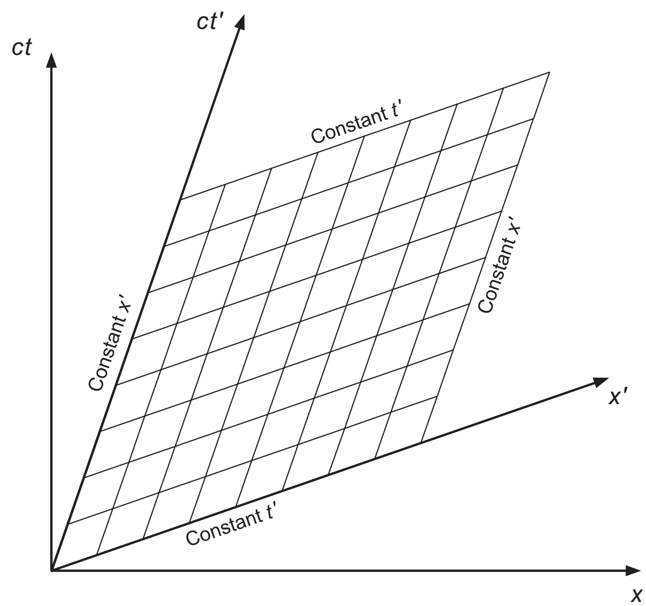
The adjective "proper" in relativity generally denotes *a quantity measured in the rest frame of the object.*

Fundamental measurement issues:

- Distances must be measured between spacetime points *at the same time.*
- Elapsed times must be measured at spacetime points *at the same place.*

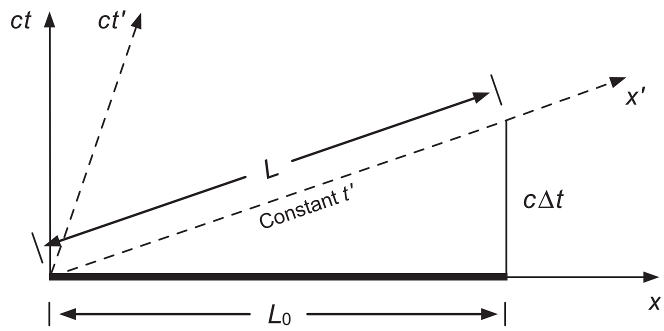
Example: For an arrow in flight its length is not generally given by the difference between the location of its tip at one time and its tail at a different time.





- The frame (t',x') is boosted by a velocity v along the positive x axis relative to the (t,x) frame. Therefore, in the primed frame the rod will have a velocity v in the negative x' direction.

- Determining the length L observed in the primed frame requires that the positions of the ends of the rod be measured *simultaneously in that frame.* The axis labeled x' corresponds to constant t' (see bottom figure above), so the distance marked as L is the length in the primed frame.

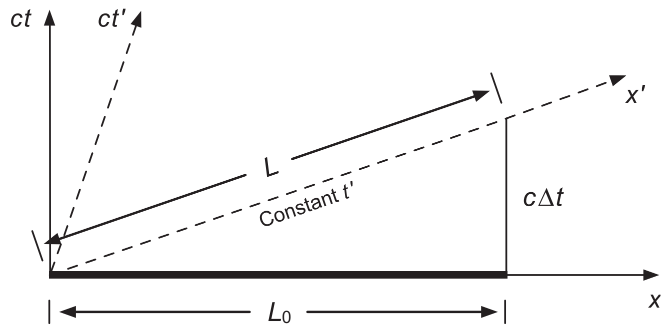


Map Projections

Mercator
(preserves angles, distorts sizes)

Source: *http://www.culturaldetective.com/worldmaps.html*

- This distance L seems longer than L_0, but this is deceiving because *we are looking at a slice of Minkowski spacetime represented on a piece of euclidean paper* (the printer was fresh out of Minkowski-space paper :).



- Much as for a Mercator projection of the globe onto a euclidean sheet of paper (which gives misleading distance information—Greenland isn't really larger than Brazil, and Africa has 14 times the area of Greenland), *we must trust the metric to determine the correct distance in a space*.

- From the Minkowski indefinite-metric line element

$$ds^2 = -c^2dt^2 + dx^2.$$

and the triangle in the figure above *(Pythagorean theorem in Minkowski space)*,

$$L^2 = L_0^2 - (c\Delta t)^2.$$

But the equation for the x' axis gives $c\Delta t = (v/c)L_0$, from which

$$L = (L_0^2 - (c\Delta t)^2)^{1/2} = \left(L_0^2 - \left(\frac{v}{c}L_0\right)^2\right)^{1/2} = L_0(1 - v^2/c^2)^{1/2},$$

which is the *length-contraction formula of special relativity*: L is *shorter* than the proper length L_0, even though it appears to be longer in the figure.

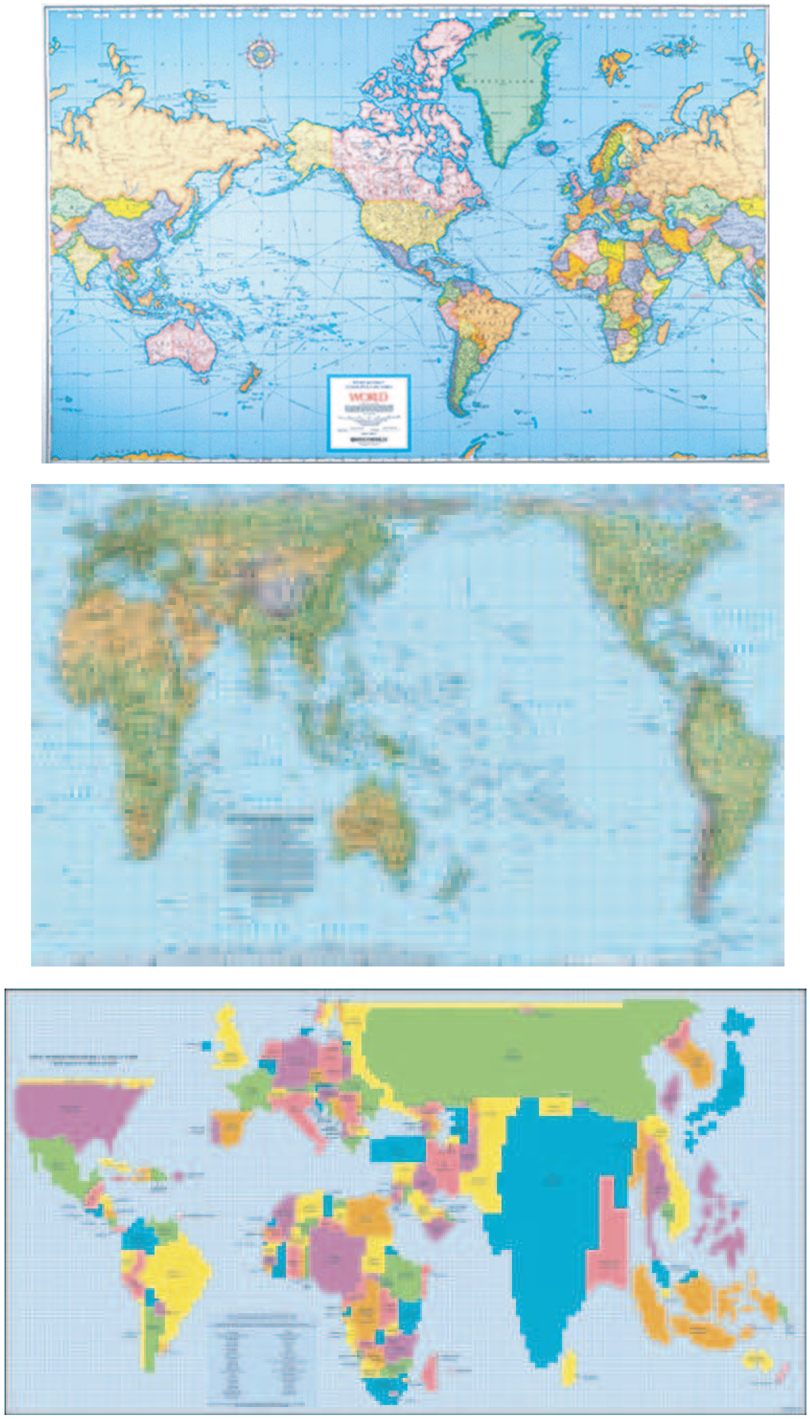


Figure 4.9: Map projections.

Invariance and Simultaneity

- In Galilean relativity, an event picks out a hyperplane of simultaneity in the spacetime diagram consisting of all events occurring at the same time as the event.
- All observers agree on what constitutes this set of simultaneous events because *Galilean relativity of simultaneity is independent of the observer*.
- *In Einstein's relativity, simultaneity depends on the observer* and hyperplanes of constant coordinate time have no invariant meaning.
- However, all observers agree on the position in spacetime of the lightcones associated with events, because *the speed of light is invariant for all observers*.

The local lightcones define an invariant spacetime structure that may be used to classify events.

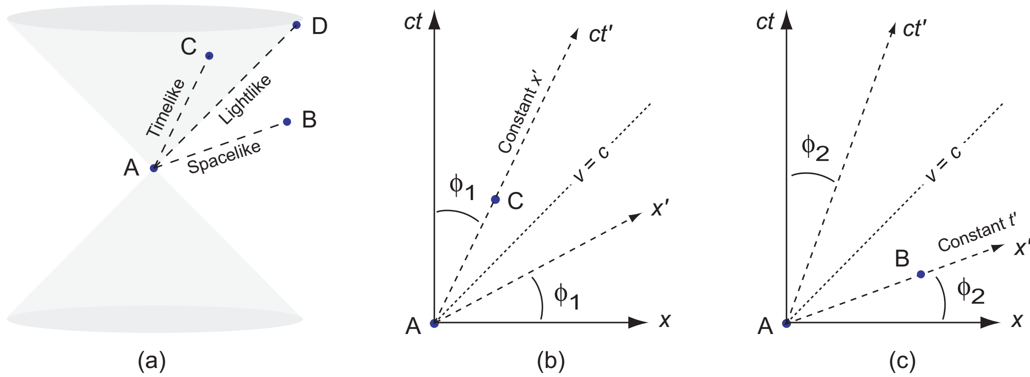
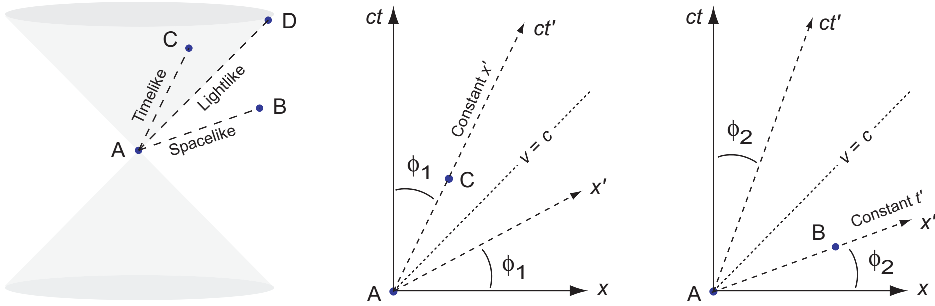


Figure 4.10: (a) Timelike, lightlike (null), and spacelike separations. (b) Lorentz transformation that brings the timelike separated points A and C of (a) into spatial congruence (they lie along a line of constant x' in the primed system). (c) Lorentz transformation that brings the spacelike separated points A and B of (a) into coincidence in time (they lie along a line of constant t' in the primed system.

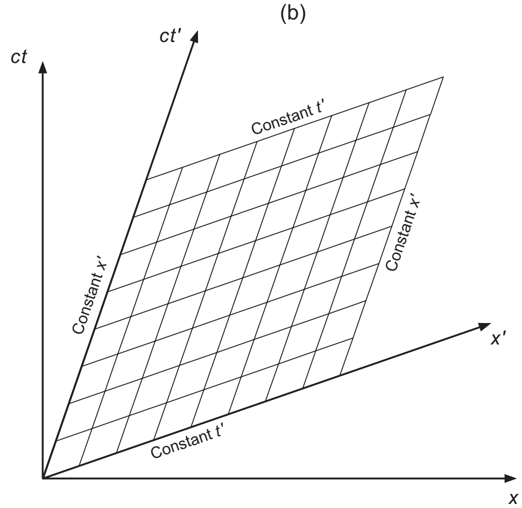
- The spacetime separation between any two events (spacetime interval) may be classified in a relativistically invariant way as

  1. timelike,
  2. lightlike,
  3. spacelike

  by constructing the lightcone at one of the points, as illustrated in Fig. 4.10(a).



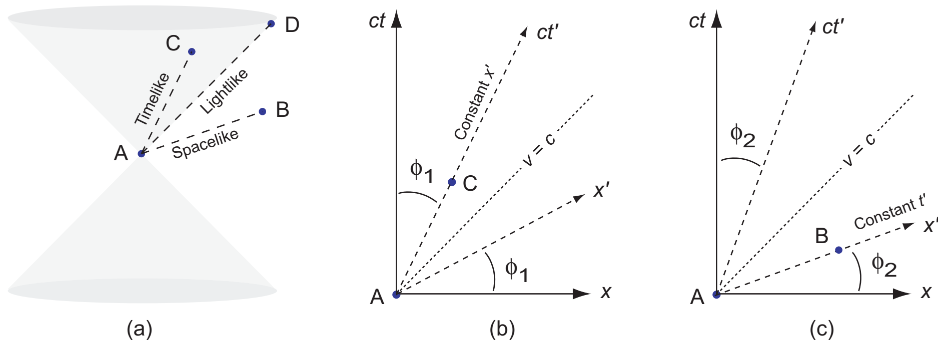
(a) (b) (c)



The geometry of the above two figures suggests another important distinction between points at spacelike separations [the line AB in Fig. (a)] and timelike separations [the line AC in Fig. (a)]:

- If two events have a *timelike separation*, a Lorentz transformation exists that can bring them into spatial congruence. Figure (b) illustrates geometrically a coordinate system (ct', x'), related to the original system by an x-axis Lorentz boost of $v/c = \tan\varphi_1$, in which A and C have the same coordinate x'.

- If two events have a *spacelike separation*, a Lorentz transformation exists that can synchronize the two points. Figure (c) illustrates an x-axis Lorentz boost by $v/c = \tan\varphi_2$ to a system in which A and B have the same time t'.



- Notice that the maximum values of φ_1 and φ_2 are limited by the $v = c$ line.
- Thus, the Lorentz transformation to bring point A into spatial congruence with point C exists only if point C lies to the left of the $v = c$ line and thus is separated by a timelike interval from point A.
- Likewise, the Lorentz transformation to synchronize point A with point B exists only if B lies to the right of the $v = c$ line, meaning that it is separated by a spacelike interval from A.