Chapter 6

Lecture: Principle of Equivalence

The general theory of relativity rests upon two principles that are in fact related:

- The principle of equivalence
- The principle of general covariance

6.1 Inertial and Gravitational Mass

1. The inertial mass is defined through Newton’s second law of motion: \( m = F/a \).

2. The gravitational mass is defined through Newton’s law of gravitation: \( m = r^2 F/GM \).

The relationship between inertial and gravitational masses was suggested by Galileo: *different objects fall at the same rate in a gravitational field*. Established to high precision in the Eötvös experiment of 1893.
If the inertial and gravitational masses differ a couple will be produced by the action on the inertial mass of the centripetal effects associated with Earth’s rotation and the balance will twist.

The result of the original Eötvös experiment was that the inertial and gravitational masses are proportional to each other with a sensitivity of one part in $10^9$ (now $10^{13}$).

**Weak Principle of Equivalence:**

$$m_{\text{inertial}} = m_{\text{grav}}.$$  

(All particles experience the same acceleration in a gravitational field, irrespective of their masses.)
6.2 The Strong Equivalence Principle

Einstein extended this idea to the modern equivalence principle (sometimes called the strong principle of equivalence).

Elevator illustrated in Fig. 6.2: unable to distinguish an acceleration of the elevator at some point in space where no gravitation fields act from the effect of a stationary elevator sitting in a planetary gravitational field.

**The Principle of Equivalence:** For an observer in free fall in a gravitational field, the results of all local experiments are independent of the magnitude of the gravitational field.
Alternative statements of Equivalence:

- All *local*, freely falling, non-rotating laboratories are fully equivalent for the performance of physical experiments. Such a laboratory is called a *local inertial frame* or *Lorentz frame*.

- In any sufficiently *local* region of spacetime, the effect of gravity can be transformed away.

- In any sufficiently *local* region of spacetime, we may construct a *local inertial system* in which the special theory of relativity is valid, even in a very strong gravitational field.

Freely falling $\rightarrow$ weightlessness: any experiment would reveal any object in the laboratory to be weightless.
6.3 Deflection of Light in a Gravitational Field

By applying the principle of equivalence, Einstein was able to obtain important results of the general theory of relativity even before he could solve the corresponding field equations. Consider Fig. 6.3.

**INTERIOR OBSERVER:** Equivalence → we may transform away the effect of gravity. Observer in the interior is unaware of any gravitational field and sees light travel in a straight line.

**EXTERIOR OBSERVER:** Aware of the gravitational field (sees the elevator falling!). The spot at which the light strikes the right wall has fallen by the same amount as the elevator.

**RECONCILE (observers must agree on laws of physics):** Light must follow a curved path as it propagates in a gravitational field.
6.3.1 Strength of the Gravitational Field

Bending of the light in the gravitational field may be characterized by a radius of curvature (see Exercise 6.1)

\[ r_c = \frac{c^2}{g}, \]

Strength of gravitational field at the surface of a gravitating object such as a star quantified through:

\[ \frac{R}{r_c} = \frac{GM}{Rc^2} = \frac{\text{Actual radius}}{\text{Light curvature radius}}, \]

where \( g = \frac{GM}{R^2} \) has been used. If \( GM/Rc^2 \ll 1 \), the field is weak. Also may be expressed as

\[ \frac{R}{r_c} = \frac{GM}{Rc^2} \cdot \frac{m}{mc^2} = \frac{GMm/R}{mc^2} = \frac{E_g}{E_0} = \frac{\text{Gravitational energy}}{\text{Rest-mass energy}}, \]

- **Weak Field:** gravitational energy of a test particle of mass \( m \) is much less than its rest mass energy (Newtonian gravity valid).

- **EXAMPLE:** White dwarf Sirius B has \( \rho \sim 10^6 \text{ g cm}^{-3} \), which gives \( R/r_c \sim 10^{-4} \). *Even at the surface of a white dwarf the gravitational field is weak on the natural scale set by light curvature.*

- **EXAMPLE:** At the surface of a neutron star or at the event horizon of a black hole, gravitational curvature radius \( \sim \) actual radius \( \to \) general relativity.
6.4 The Gravitational Redshift

INTERNAL OBSERVER: Free fall, unaware of gravity, \( v = v_0 \).

EXTERNAL OBSERVER: Aware of gravity (sees the elevator falling!).

When light reaches the ceiling a time \( t = h/c \) has elapsed and the elevator has accelerated to a velocity \( v = gt = gh/c \).

Doppler shift:

\[
\frac{\Delta v}{v} = \frac{v}{c} = \frac{gh}{c} = \frac{GMh}{R^2c^2} \quad \text{since} \quad g = \frac{GM}{R^2}
\]

RECONCILE: To avoid contradiction, must be a red shift produced by the gravitational field that exactly cancels the blue shift (produced by velocity, not gravity). Photons propagating upward in a gravitational field for a short distance experience a redshift given by above equation.
Total Redshift in a Gravitational Field

Integrated redshift assuming relatively weak gravity

\[ \int_{\nu_0}^{\nu_s} \frac{d\nu}{\nu} = - \int_{R}^{s} \frac{GM}{r^2c^2} dr, \]

**Exercise:** Integrating, exponentiating both sides, and then expanding the right-side exponential (weak-field assumption) gives at radius \( s \),

\[ \frac{\nu_s}{\nu_0} \approx 1 - \frac{GM}{c^2} \left( \frac{1}{R} - \frac{1}{s} \right). \]

For a distant observer \( s \to \infty \) and (since the field is assumed weak)

\[ \frac{\nu_\infty}{\nu_0} \approx 1 - \frac{GM}{Rc^2} \quad \rightarrow \quad \frac{\nu_0}{\nu_\infty} = \left( 1 - \frac{GM}{Rc^2} \right)^{-1} \approx 1 + \frac{GM}{Rc^2} \quad (GM << Rc^2) \]

The corresponding gravitational redshift \( z \) for the weak field limit is

\[ z \equiv \frac{\lambda_\infty - \lambda_0}{\lambda_0} = \frac{\nu_0}{\nu_\infty} - 1 \approx \frac{GM}{Rc^2} \quad \text{(weak field limit)} \]

where \( R \) is the radius of the star and \( M \) is its mass.

The full result valid for both strong and weak gravitational fields is

\[ 1 + z = \left( 1 - \frac{2GM}{Rc^2} \right)^{-1/2}, \]

(will derive later). Reduces to above if the field is weak.
**EXAMPLE**: for the white dwarf Sirius B

\[ R = 5.5 \times 10^8 \text{ cm} \quad M = 2.1 \times 10^{33} \text{ g} \]

Inserting these values

\[ z \approx \frac{GM}{Rc^2} \approx 2.8 \times 10^{-4} \]

Measured redshift: \( z = 3.0 \pm 0.5 \times 10^{-4} \) from displacement of spectral lines
6.4.2 Gravitational Time Dilation

Gravitational redshift may be viewed as \textit{gravitational time dilation}:

\[
\text{time} \propto \frac{1}{\text{frequency}}
\]

One period of light wave = One clock tick

\[
\frac{\Delta t_0}{\Delta t_\infty} \simeq \frac{v_\infty}{v_0} \simeq 1 - \frac{GM}{Rc^2} \quad (\text{weak field limit}),
\]

\textbf{EXAMPLE:} For surface of Sirius B

\[
\frac{\Delta t_0}{\Delta t_\infty} \simeq 1 - \frac{GM}{Rc^2} \simeq 0.99972 \quad (\sim \text{One second per hour slow})
\]

For strong fields (will derive later)

\[
\frac{\Delta t_0}{\Delta t_\infty} = \sqrt{1 - \frac{2GM}{Rc^2}}
\]

Reduces to above when the second term under the radical is small.

Purely gravitational effect, independent of any special relativistic time dilation due to relative motion between source and observer (GPS corrections require both).
6.5  Riemanian Spaces

We cannot set up a global Cartesian coordinate system on a curved surface but if the metric takes the form (2D for illustration)

\[ ds^2 = a(x,y)dx^2 + 2b(x,y)dxdy + c(x,y)dy^2 \]

the corresponding space is *locally Euclidean*:

- At any point a locally-valid Cartesian coordinate system may be constructed
- The circumference of a circle: \( C = 2\pi r + \text{higher-order terms} \)
- The sum of the angles of a triangle: \( \pi + \text{higher-order terms} \)

Such a space is termed a **Riemannian space**, with a corresponding **Riemannian metric**.

**COUNTER-EXAMPLE:**

\[ ds^2 = (dx^4 + dy^4)^{1/2} \]  \( \text{(Finsler metric)} \)

The geometry is not Euclidean, even locally.
**EXPLICIT CONSTRUCTION:**

If for

$$ds^2 = a(x,y)dx^2 + 2b(x,y)dxdy + c(x,y)dy^2$$

we define at an arbitrary point $P_0 = (x_0, y_0)$

$$X = a_0^{1/2}x + \frac{b_0}{a_0^{1/2}}y \quad Y = \left(c_0 - \frac{b_0^2}{a_0}\right)^{1/2}y$$

where $a_0 \equiv a(P_0) \ldots$, then near $P_0$

$$ds^2 = dX^2 + dY^2 \quad \text{(Pythagoras)}$$

and geometry is *locally Euclidean* around the arbitrary point $P_0$.

This demonstrates explicitly in two dimensions the locally Euclidean nature of the Riemannian metric. Conversely, if the metric is Euclidean around an arbitrary local point $P_0$ the space is necessarily Riemannian.
The preceding considerations suggest

\[ \text{Equivalence Principle} \leftrightarrow \text{Riemannian Geometry} \]

which in turn implies

\[ \text{Gravitation} \leftrightarrow \text{Riemannian Geometry} \]

This relationship foreshadowed in the work of Gauss and Riemann during the 19th century:

- **Gauss:** all *inner (intrinsic) properties* of a curved surface are described by the derivatives $\frac{\partial \xi^\alpha}{\partial x^\mu}$ of the functions $\xi^\alpha(x)$ implementing the transformation between a general coordinate system $x^\mu$ and a local Cartesian coordinate system $\xi^\alpha(x)$.

- Because of equivalence, all effects of a gravitational field will be describable in terms of the derivatives $\frac{\partial \xi^\alpha}{\partial x^\mu}$ of the functions $\xi^\alpha(x)$ that define the transformation between a general coordinate system $x^\mu$ and a local inertial coordinate system that may (by the principle of equivalence) be constructed at any point in spacetime.

Thus, the principle of equivalence will find its natural mathematical expression in Riemannian geometry.
6.6 *Local Inertial Systems*

*Equivalence:* Elevator occupants on opposite sides of the Earth may replace the gravitational field by a local acceleration (Fig. 6.5).

*No Contradiction:* The two elevator occupants in this case cannot be in the same local inertial frame.

*Operational Definition of Local:* Tidal effects (Fig. 6.6) are negligible. (Later: quantitative definition in terms of curvature.)
6.7 The Path to a Covariant Gravitational Theory

- **Einstein**: Inhomogeneity of gravitational field caused by inhomogeneity of gravitating matter → spacetime is curved, with curvature related to the distribution of matter (and energy, . . .).

- **Equivalence Principle**: Spacetime is a patchwork of *locally flat frames* meshed smoothly to describe a curved space (Fig. 6.7).

- **Key to relating spacetime curvature and matter distribution**: The connection between equivalence principle and Riemannian geometry.

- **Some Sewing Required**: Local Euclidean patches (where gravity can be banished and special relativity holds) must be “stitched together” smoothly to form a Riemannian manifold (Fig. 6.7).

- **General Relativity**: A “stitching together” prescription. Find a (unique, spacetime-dependent) Riemannian metric determined by a nonlinear relation among mass, curvature, density.