9 The Theory of Special Relativity

Assign: Read Chapter 4 of Carrol and Ostlie (2006)

Newtonian physics is a quantitative description of Nature except under three circumstances:

1. In the realm of the very small (sub-atomic), deterministic Newtonian physics must be replace by *quantum mechanics,* as we have already seen.
2. As we shall discuss in this section, if velocities approach the speed of light Newtonian physics must be replace by Einstein's *special theory of relativity.*
3. If the strength is gravitational fields is very large (much larger than that of the Earth), Newtonian physics must be replaced by Einstein's *general theory of relativity.*

> The special theory of relativity is a special case of the general theory of relativity, valid if the strengths of all gravitational fields are weak. Since the general theory contains the special theory as a special case, it is also valid when velocities are comparable to that of light.

What if these conditions are combined? Generally, we find that

1. If the dimensions of the system are very small *and* the velocities are comparable to light velocity, Newtonian physics must be replaced by a wedding of quantum mechanics and special relativity called *relativistic quantum field theory.*
2. If the dimensions of the system are very small *and* the gravitational fields are very strong on these small dimensions, Newtonian mechanics must be replace by a *theory of quantum gravity* that somehow combines quantum mechanics and general relativity. No one has solved this problem yet.

In this section we introduce the basic ideas of special relativity. In a later section we shall address the more difficult topic of general relativity.

9.1 Postulates of the Special Theory

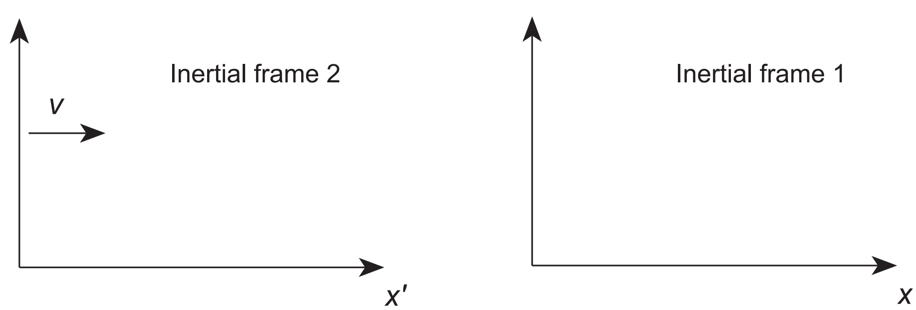
The special theory of relativity is built on two deceptively simple postulates:

1. Observers in different inertial frames should agree on the laws of physics (*principle of relativity*).
2. The speed of light in vacuum is a constant, independent of which inertial frame from which it is measured.

The concept of an *inertial frame* is central to these postulates. An inertial frame is a frame of reference (coordinate system) in which Newton's first law holds.

> The operational test of whether one is in an inertial frame is to throw an object. If it travels in a straight line then you are in an inertial frame. If the path is curved, you are not observing from an inertial frame (that is, there are external forces acting).

Two inertial frames therefore move with constant velocity with respect to each other:

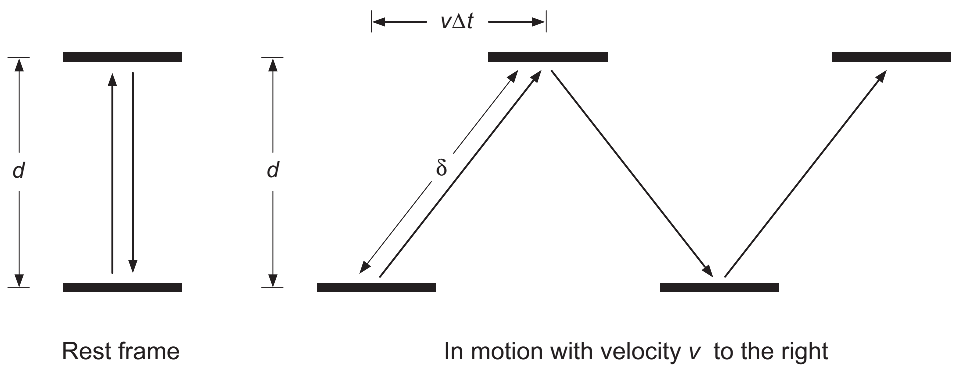


The transformation between two inertial frames is called a *Lorentz transformation.*

> Generally the Lorentz transformation involves a change in orientation angle and a change in velocity. The special case of a transformation between frames with the same orientation but different velocities is called a *Lorentz boost.*

9.2 Time Dilation

To explore the nature of space and time we need something to measure time (clocks) and something to measure distance (rulers). Consider the following schematic clock called a *light clock.*



A light ray reflects between two parallel mirrored surfaces. Each time the light strikes one of the surfaces, the clock ticks. Consider two cases

1. The observer is at rest with respect to the clock (left part of figure). We say that the observation is in the *rest frame* of the clock.
2. The observer is moving with a speed v to the left with respect to the clock, so she sees the clock moving to the right relative to her (right side of figure). Thus, this observer is in a *different inertial frame* than the one in the rest frame of the clock.

We now analyze what the two observers see. In the rest frame of the clock the time observed between ticks is

$$\Delta t' = \frac{d}{c}. \tag{43}$$

But for the second observer moving with respect to the clock, the distance that the light has to travel between ticks of the clock is not d but (from the right triangles)

$$\delta = \sqrt{d^2 + (v\Delta t)^2}.$$

Thus, for the *same clock* the observer in motion relative to the clock sees a time interval between clicks

$$\Delta t = \frac{\sqrt{d^2 + (v\Delta t)^2}}{c}.$$

Squaring both sides and solving for Δt gives

$$c^2(\Delta t)^2 = d^2 + v^2(\Delta t)^2 \quad \to \quad (c^2 - v^2)(\Delta t)^2 = d^2$$
$$\to \quad (\Delta t)^2 = \frac{d^2}{c^2(1 - v^2/c^2)} \quad \to \quad \Delta t = \frac{d}{c\sqrt{1 - v^2/c^2}}.$$

Finally, substituting $d = c\Delta t'$ from Eq. (43) above gives

$$\Delta t = \frac{c\Delta t'}{c\sqrt{1 - v^2/c^2}} = \frac{\Delta t'}{\sqrt{1 - v^2/c^2}}. \tag{44}$$

Thus, the clocks (or any physical measure of time) do not run at the same rate in the two systems. Letting

$\tau_0 \equiv \Delta t'$ (Time interval in rest frame)

$\tau \equiv \Delta t$ (Time interval in moving frame)

This may be expressed as

$$\tau = \frac{\tau_0}{\sqrt{1 - v^2/c^2}} = \gamma \tau_0, \tag{45}$$

where the *Lorentz* γ*-factor* is defined by

$$\gamma \equiv \frac{1}{\sqrt{1 - v^2/c^2}} \qquad (\gamma \geq 1). \tag{46}$$

This effect is called *time dilation*, because the time τ measured in the moving frame is always *greater* than the time τ_0 measured in the rest frame of the clock. The time interval τ_0 measured in the rest frame of the clock is called the *proper time.*

Example: Time dilation effects

Consider a car going at 100 km hr^{-1}, observing a clock beside the road. The relative velocity in units of c is

$$\frac{v}{c} = \frac{(100\ \text{km hr}^{-1})(1\ \text{hr}/3600\ \text{s})}{3 \times 10^5\ \text{km s}^{-1}} = 9.3 \times 10^{-8}.$$

Then the difference in measured time intervals (length of clock ticks) between the rest frame of the car τ and the rest frame of the clock τ_0 is

$$\frac{\tau}{\tau_0} = \frac{1}{\sqrt{1 - v^2/c^2}} = \frac{1}{\sqrt{1 - (9.3 \times 10^{-8})^2}} = \frac{1}{\sqrt{1 - 8.6 \times 10^{-15}}} \simeq 1.$$

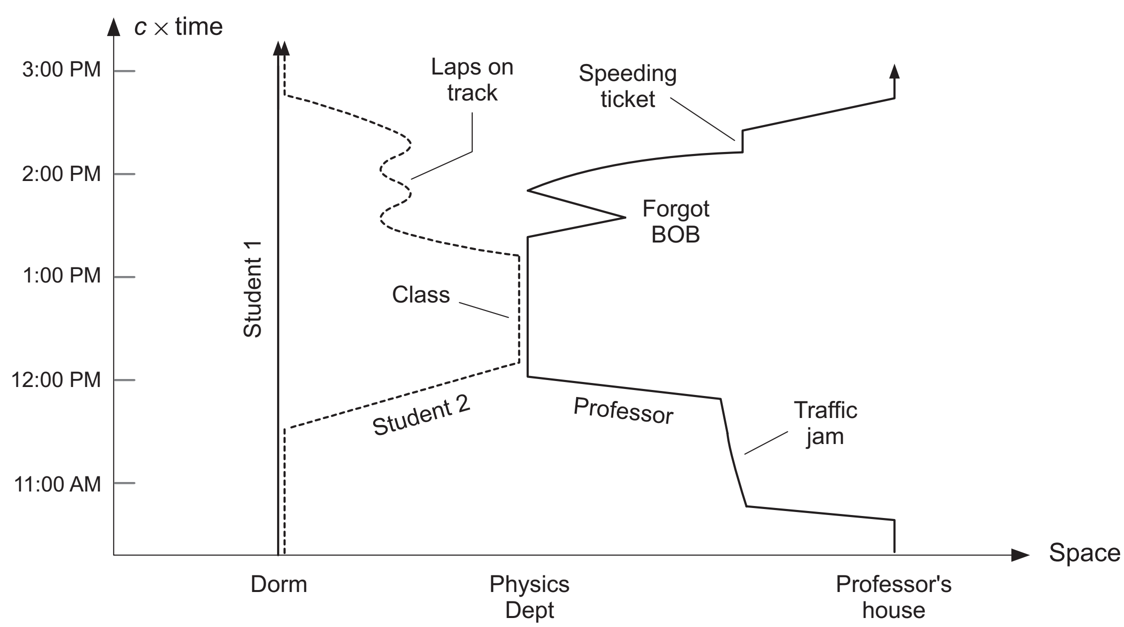
Thus the time dilation effect is negligible. But now suppose a starship coming down the same road at $v = 0.99c$. Then $v^2/c^2 = (0.99)^2 = 0.98$, and we obtain

$$\frac{\tau}{\tau_0} = \frac{1}{\sqrt{1 - v^2/c^2}} = \frac{1}{\sqrt{1 - 0.98}} = \frac{1}{\sqrt{0.02}} \simeq 7.1.$$

So from the rest frame of the starship the clock appears to be ticking about 7 times slower than for an observer on the side of the road (rest frame of the clock).

9.2.1 Spacetime diagram

In special relativity it is convenient to introduce a *spacetime diagram* for which the horizontal axis is space (distance) and the vertical axis is ct (which has dimensions of distance). For example,

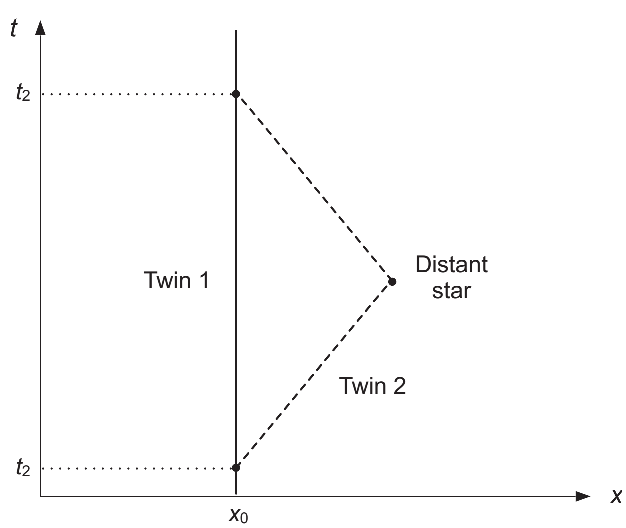


(Note that to be realistic the scale of the vertical and horizontal axes would be very different in this example.)

> The lines followed by objects are called *world lines* and individual spacetime points are called *events*.

9.2.2 The twin paradox

Different paths in spacetime generally correspond to different rates of time flow. A famous example is the *twin paradox* (not really a paradox, but that is the historical name). Twins are born on Earth. One becomes an astronaut and goes off to a distant star in a spaceship at a speed $v \simeq c$, turns around, and comes back at the same speed. What are the ages of the twins when they are reunited? The problem is illustrated in the following spacetime diagram.



From the diagram, the twins have not followed the same spacetime paths, so they may be expected to age differently because of time dilation.

Example: A Tale of Two Twins

Let's take as a specific example of the twin paradox a trip to Sirius (about 8 light years away), initiated when the twins are 25, with $v = 0.99c$ for the speed of the spacecraft both going to and coming from Sirius. Since the speed is almost lightspeed, it takes about 8 years to get to Sirius and 8 years to return, so the twin on Earth will see about $2 \times 8 = 16$ years elapse from the departure and return of the astronaut twin; the twin who stayed on Earth will be $25 + 16 = 41$ years old on the date of return. From the time dilation formula for $v/c = 0.99$ we have already computed in an earlier example that

$$\frac{\tau}{\tau_0} \simeq 7.1.$$

Therefore, for the astronaut twin time will pass 7.1 times slower, and he will age only $16/7.1 = 2.3$ years during the trip. Upon return the astronaut twin will be $25 + 2.3 = 27.3$ years old but the earthbound twin will be 41 years old!

9.3 Space Contraction

By similar arguments as we made for deriving the time dilation formula, we can show that measured *distances* depend on the relative velocity. For example, if a rod has length ℓ_0 in its rest frame, its length ℓ measured in a frame moving with a speed v (parallel to the rod) relative to the rod's rest frame is

$$\ell = \ell_0 \sqrt{1 - v^2/c^2} = \frac{\ell_0}{\gamma} \qquad \gamma \equiv \frac{1}{\sqrt{1 - v^2/c^2}}. \tag{47}$$

Thus, the length of the rod is different as measured in the two frames.

This effect is called *space contraction* because the length ℓ measured in the moving frame is *less than* the length ℓ_0 measured in the rest frame. The length of the rod ℓ_0 measured in its own rest frame is termed its *proper length.*

Example: Length contraction

Suppose a track of length 100 m. If a sprinter runs down the track with a speed of 10 m s^{-1}, how long does the track appear to the sprinter? We have

$$\frac{v}{c} = \frac{10 \text{ m s}^{-1}}{3 \times 10^8 \text{ m s}^{-1}} = 3.3 \times 10^{-8},$$

so the Lorentz γ-factor is

$$\gamma = \frac{1}{\sqrt{1 - v^2/c^2}} = \frac{1}{\sqrt{1 - 1.1 \times 10^{-15}}} \simeq 1,$$

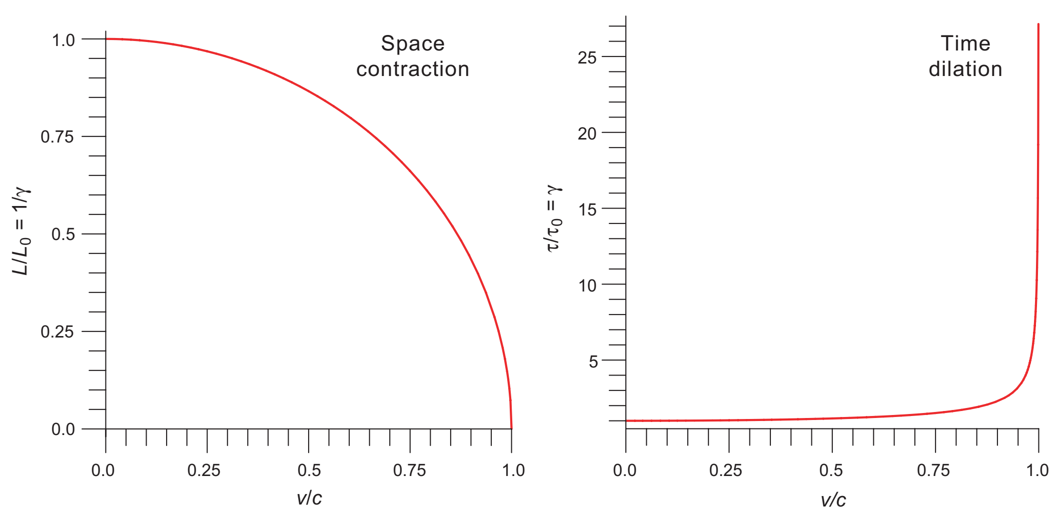
and the track appears to be 100 meters long. Now suppose the sprinter could run at a speed of $v/c = 0.5$. Then

$$\gamma = \frac{1}{\sqrt{1 - v^2/c^2}} = \frac{1}{\sqrt{1 - 0.25}} = 1.15,$$

and the track would appear to be contracted to a length of

$$\ell = \frac{\ell_0}{\gamma} = \frac{100 \text{ m}}{1.15} = 86.6 \text{ m}.$$

The behavior of space contraction and time dilation with v/c is summarized in the following figure:



9.4 Mass and Energy

Another important result of special relativity is that the total energy E of a particle is related to its mass (measured at rest) m and the magnitude of its momentum p by

$$E = \sqrt{p^2c^2 + m^2c^4}. \tag{48}$$

If the particle is at rest, $p = 0$ and we obtain the most famous equation in science:

$$E = mc^2. \tag{49}$$

> Relativity implies that mass and energy are really two aspects of the same thing, with the exchange rate between mass and energy being c^2. Because c^2 is a very large number, even a particle at rest contains a very large amount of energy, by virtue of its mass.

Example: Conversion of mass to energy in the Sun

The luminosity of the Sun is $L_{\odot} = 3.839 \times 10^{26}$ J s^{-1}. Assuming this to be due ultimately to conversion of mass to energy by thermonuclear processes in its core (as we shall discuss later), how much mass must be converted to energy each second to produce this luminosity?

In one second the Sun produces 3.839×10^{26} J of energy, so

$$m = \frac{E}{c^2} = \frac{3.839 \times 10^{26}\ \mathrm{kg\,m^2s^{-2}}}{(3 \times 10^8\ \mathrm{m\,s^{-1}})^2} = 4.27 \times 10^9\ \mathrm{kg}$$

of mass must be converted each second to produce the solar luminosity.

9.5 The Newtonian Limit

In the limit of low velocities the results of special relativity reduce to those of Newtonian mechanics. We have already seen this for time dilation and space contraction, where if $v/c \rightarrow 0$, the Lorentz $\gamma \rightarrow 1$ and the time dilation and space contraction become negligible.

For the relativistic energy expression

$$\begin{aligned} E &= \left(p^2c^2 + m^2c^4\right)^{1/2} \\ &= \left(1 + \frac{p^2c^2}{m^2c^4}\right)^{1/2} mc^2 \\ &\simeq \left(1 + \tfrac{1}{2}\frac{p^2c^2}{m^2c^4}\right) mc^2 \\ &= mc^2 + \frac{p^2c^2mc^2}{2m^2c^4} \\ &= mc^2 + \frac{p^2}{2m}, \end{aligned}$$

where in the third line we've used the binomial approximation

$$(1+\alpha)^n \simeq 1 + n\alpha,$$

which is valid for small α, and

$$\frac{p^2}{2m} = \tfrac{1}{2}mv^2$$

is the Newtonian kinetic energy.

9.6 General Relativity: An Introduction

General relativity (GR) is a much more complicated theory than special relativity. At this point we will give a qualitative introduction to the basic ideas. In the second semester, we shall have more to say about general relativity.

9.6.1 Generalization of the principle of relativity

The basic idea of GR is to remove the restriction in special relativity and require that *the laws of physics are invariant under transformations between any reference frames, whether they are inertial or not.*

In particular, this means that we may deal with transformations between reference frames that are accelerating with respect to each other. This will allow us to gravitational accelerations and forces: *general relativity is a new theory of gravitation that replaces Newton's law of gravitation.*

9.6.2 The principle of equivalence

The mathematics of GR is much more complicated than that for special relativity, but the conceptual foundation of GR is a *principle of equivalence,* which can be stated non-mathematically.

The starting point is the observation that in Newtonian physics there are two operational definitions of mass:

1. The *inertial mass* is defined through Newton's second law of motion:

   $$m_{\text{inertial}} = \frac{F}{a},$$

   where m is the inertial mass, F is the magnitude of the force, and a is the magnitude of the acceleration.

2. The *gravitational mass* is defined through Newton's law of gravitation:

   $$m_{\text{grav}} = \frac{r^2 F}{GM},$$

   where m is the gravitational mass, r is the separation of the mass from a gravitating sphere like the Earth, F is the magnitude of the gravitational force, M is the mass of the gravitating sphere, and G is the gravitational constant.

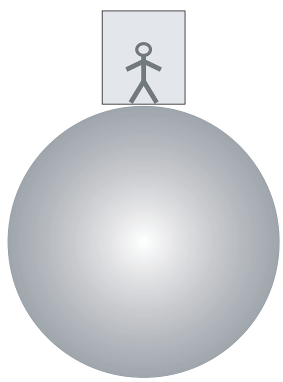
Notice the asymmetry of these definitions: the first applies for *any* force; the second applies only for a particular force, the gravitational force.

These masses for an object need not be equivalent, but the *weak equivalence principle* is that

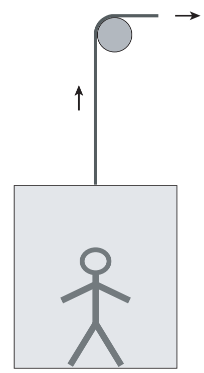
$$m_{\text{inertial}} = m_{\text{grav}}$$

for any object.

Einstein extended this idea to what we now call the *strong equivalence principle* by using a series of thought experiments that can be illustrated by an elevator.



Stationary elevator in a gravitational field at the surface of a planet



Elevator accelerated in interstellar space far from gravitating masses

The Strong Equivalence Principle

For an observer in free fall in a gravitational field, the results of all local experiments are independent of the magnitude of the gravitational field.

Einstein was able to use the (strong) equivalence principle to solve important problems in general relavity, even before he could solve the complicated equations of general relativity.

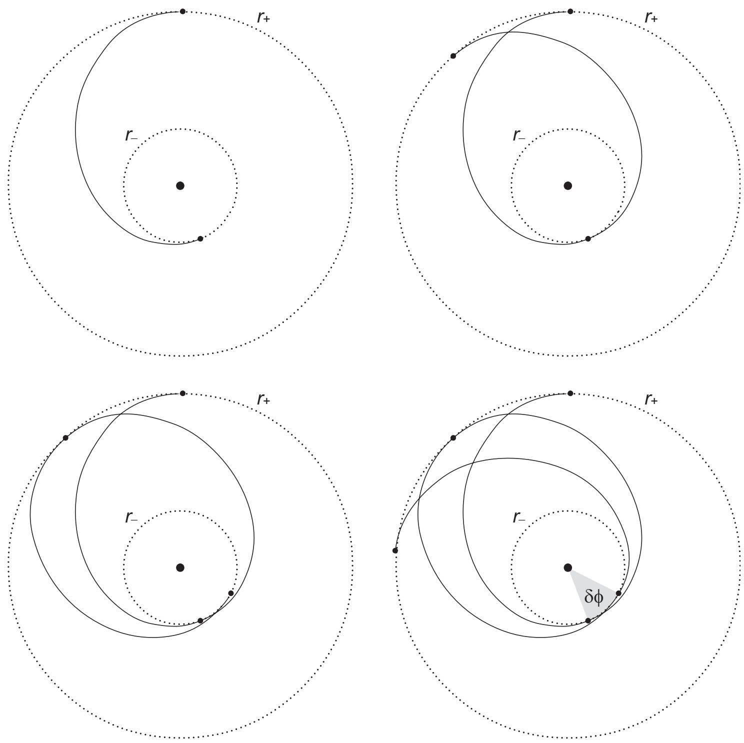
> Another way to state the principle of equivalence is that in any small enough region in spacetime it is possible to formulate the equations governing physical laws such that gravitation can be neglected. This means that the special theory of relativity is valid for that particular situation, and this in turn allows a number of things to be deduced because the solution of the equations for the special theory of relativity is much simpler than that for the general theory of relativity.

9.6.3 Tests of the general theory of relativity

As long as gravitational fields are weak and velocities are low, general relavity makes the same predictions about gravity as Newton's theory. There are several crucial points where the two theories make different predictions that are testable:

1. **Precession of gravitational orbits**

   General relativity predicts deviations from Kepler ellipses, which don't close on themselves. This was first tested on the orbit of Mercury, which shows an anomalous precession of its perhelion (point of closest approach) of $43''$ per century, exactly as predicted by GR.



2. **Gravitational redshift**

   General relavity predicts that a gravitational field shifts light propagating upward in it to longer wavelengths. (This is also equivalent to a gravitational time dilation.) This can be tested even in the weak gravitional field of the Earth and the measured redshift is found to be that predicted by GR.

3. **Deflection of light in a gravitational field**

   General relavity predicts that light is deflected by a gravitational field. This can be tested in a total solar eclipse and the amount of observed deflection is that predicted by GR.

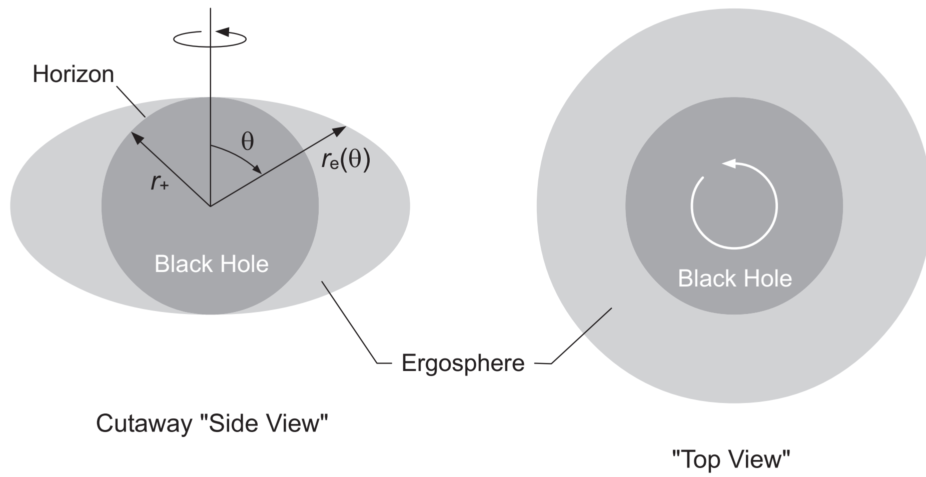
4. **Gravitational waves should exist**

   Gravitational waves have not yet been observed directly because they are very difficult to detect, but they have been observed indirectly in the Binary Pulsar.

5. **Frame dragging**

   General relavity predicts that rotating gravitational fields drag spacetime with them in a swirling motion called *frame dragging*. This effect has been observed in careful analysis of satellite data, and a more precise experiment (Gravity Probe B) is currently collecting data to test it more rigorously.

An extreme example of frame dragging: the ergosphere of a Kerr black hole



In the ergosphere one is still outside the event horizon but the frame dragging is so severe that speeds exceeding that of light would be required to counteract it. Therefore, no stationary observer is possible in the ergosphere because no matter how much rocket power is applied, the observer will be dragged around by the whirling spacetime.

9.6.4 Curvature of spacetime

According to general relativity, space and time can be curved, and it is this curvature that corresponds to the gravitational field. The stronger the curvature, the stronger the gravitational field.

> Thus Einstein reduced gravitation to the geometry of a 4-dimensional spactime continuum. Einstein showed that mass caused space to curve and that objects travelling in that curved space have their paths deflected, exactly as if a force had acted on them.