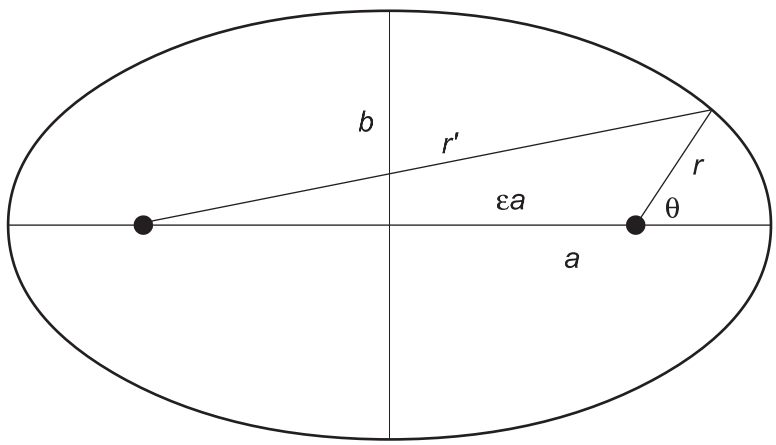
3 Celestial Mechanics

Assign: Read Chapter 2 of Carrol and Ostlie (2006)

3.1 Kepler's Laws

- OJTA: 3. The Copernican Revolution/Kepler
  - (4) Kepler's First Law



$$r + r' = 2a \qquad b^2 = a^2(1 - \epsilon^2) \tag{1}$$

$$r = \frac{a(1-\epsilon^2)}{1 + \epsilon\cos\theta} \qquad \text{Area} = \pi a b \tag{2}$$

Example: For Mars

$$a = 1.5237 \text{ AU} \qquad \epsilon = 0.0934$$

At perihelion $\theta = 0°$ and

$$r = \frac{a(1-\epsilon^2)}{1+\epsilon\cos\theta} = \frac{(1.5237 \text{ AU})(1 - 0.0934^2)}{1 + 0.0934\cos(0°)} = 1.3814 \text{ AU}$$

At aphelion $\theta = 180°$ and

$$r = \frac{(1.5237 \text{ AU})(1 - 0.0934^2)}{1 + 0.0934\cos(180°)} = 1.6660 \text{ AU}$$

Difference = 19%

– (5) Kepler's Second Law
– (6) Kepler's Third Law

$$P^2(\text{yr}) = a^3(\text{AU})$$

Example: For Venus, $a = 0.7233$ AU. Then the period is

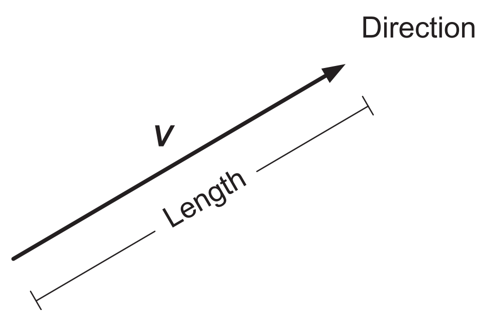
$$P = a^{3/2} = (0.7233)^{3/2} = 0.6151 \text{ yr} = 224.7 \text{ d}$$

3.2 Galileo

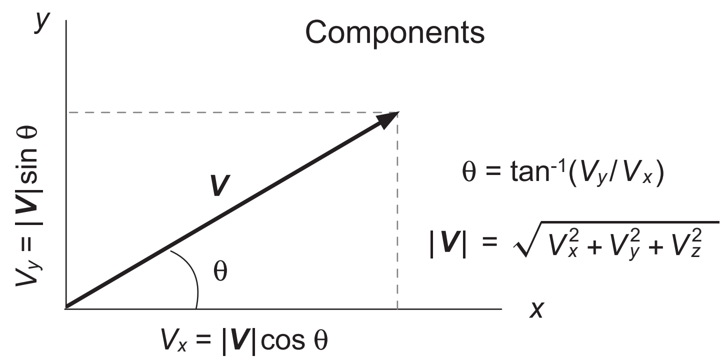
- OJTA: 4. The Modern Synthesis/Galileo
  – (3) New Telescopic Observations
  – (4) Inertia

3.3 Mathematical Interlude: Vectors

Vectors are objects that have a magnitude (length) and a direction.



Therefore, they require more than one number to specify them. In contrast a *scalar* is specified by only one number. One way to specifiy a vector is to give its components:



The direction of a vector can be specified by orientation angles (one in two dimensions and two in three dimensions). Its length is given by the Pythagorean theorem in terms of its components. For a 3-dimensional vector,

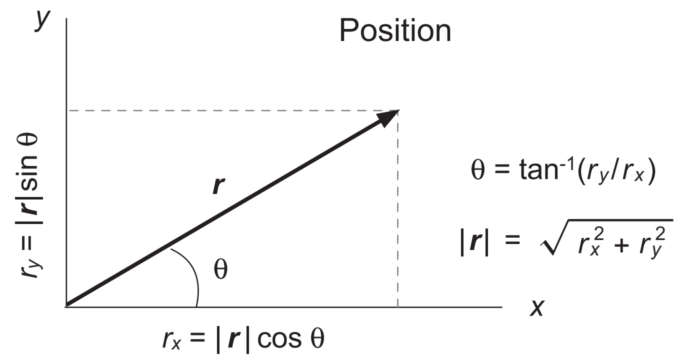
$$|V| = \sqrt{V_x^2 + V_y^2 + V_x^2}.$$

The components are related to the angles by basic trigonometry. In the 2-D example above,

$$\theta = \tan^{-1}\left(\frac{V_y}{V_x}\right)$$

Some important vector quantities include the position, velocity, momentum, and acceleration of objects. Let us illustrate in 2-D for simplicity.

Position: The position of a point can be specified by a vector $\boldsymbol{r}$

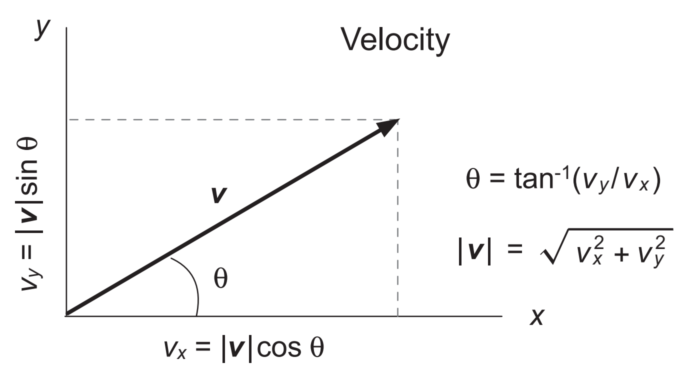


where

$$r_x = r\cos\theta \qquad r_y = r\sin\theta \qquad r \equiv |\boldsymbol{r}| = \sqrt{r_x^2 + r_y^2}.$$

Velocity: The velocity vector can be defined in terms of the time derivative of the position vector,

$$\boldsymbol{v} \equiv \frac{d\boldsymbol{r}}{dt} \simeq \frac{\Delta\boldsymbol{r}}{\Delta t} = \frac{\boldsymbol{r}_1 - \boldsymbol{r}_2}{t_1 - t_2}$$



The standard units of velocity in the SI system are $\mathrm{m\,s^{-1}}$.

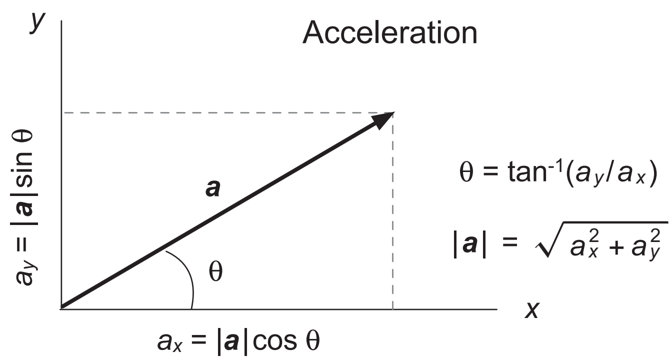
Momentum: The momentum vector is defined to be the mass m times the velocity vector,

$$\boldsymbol{p} \equiv m\boldsymbol{v} = m\frac{d\boldsymbol{r}}{dt}.$$

Its standard units are $\mathrm{kg\,m\,s^{-1}}$.

Acceleration: The acceleration vector $\boldsymbol{a}$ is defined to be the time derivative of the velocity vector,

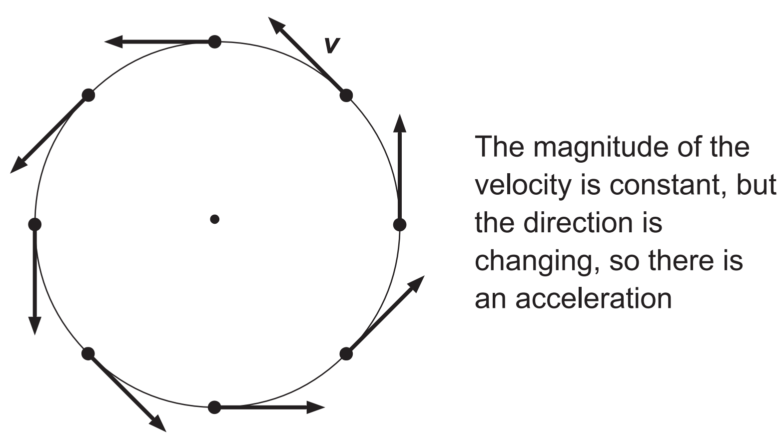
$$\boldsymbol{a} \equiv \frac{d\boldsymbol{v}}{dt} = \frac{d^2\boldsymbol{r}}{dt^2}.$$



The standard units of acceleration in the SI system are $\mathrm{m\,s^{-2}}$.

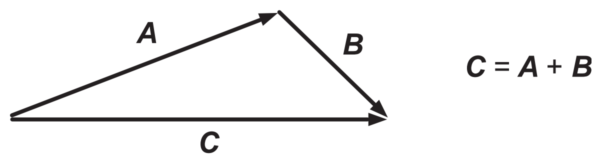
Example: Uniform circular motion

Consider the case of uniform circular motion

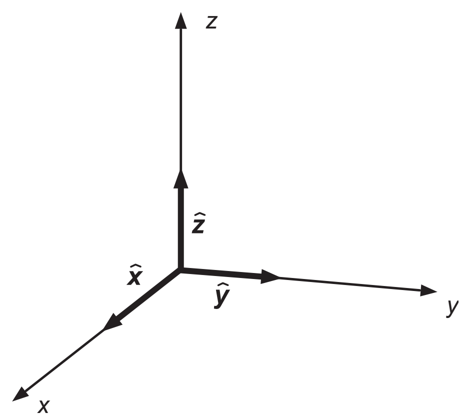


Is this accelerated motion? Yes, because there is a continuous change in the direction of the velocity, even though its magnitude is constant.

Addition of vectors: Vectors can be added graphically by a head–to–tail rule:



Unit vectors: It is convenient to define unit vectors that point along the coordinate system axis and have unit length. For cartesian coordinates,



In terms of components, we can write for a vector $\boldsymbol{A}$

$$\boldsymbol{A} = A_x \hat{\boldsymbol{x}} + A_y \hat{\boldsymbol{y}} + A_z \hat{\boldsymbol{z}}.$$

Scalar product of vectors: There are two kinds of vector products of interest to us. The *scalar product* of two vectors $\boldsymbol{A}$ and $\boldsymbol{B}$ is a number (a scalar), defined by

$$\boldsymbol{A} \cdot \boldsymbol{B} \equiv |\boldsymbol{A}||\boldsymbol{B}| \cos\theta = AB \cos\theta, \tag{3}$$

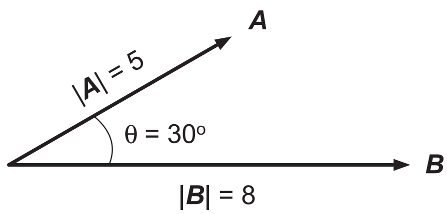
where θ is the angle between the two vectors. The order does not matter in the scalar product: $\boldsymbol{A} \cdot \boldsymbol{B} = \boldsymbol{B} \cdot \boldsymbol{A}$. That is, the scalar product *commutes*. The scalar product is often called the *dot product*.

Note some special cases of the scalar product:

1. If $\boldsymbol{A}$ and $\boldsymbol{B}$ point in the same direction, $\cos\theta = 1$ and $\boldsymbol{A}\cdot\boldsymbol{B} = AB$.
2. If $\boldsymbol{A}$ and $\boldsymbol{B}$ point in the opposite direction, $\cos\theta = -1$ and $\boldsymbol{A}\cdot\boldsymbol{B} = -AB$.
3. If $\boldsymbol{A}$ and $\boldsymbol{B}$ are perpendicular, $\cos\theta = 0$ and $\boldsymbol{A}\cdot\boldsymbol{B} = 0$.

Example: Scalar Product of Two Vectors

Consider the following two vectors



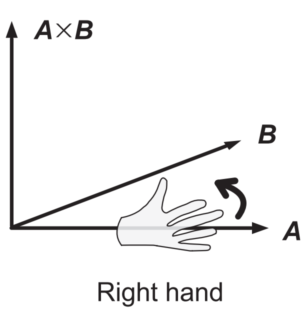
Their scalar product is

$$\boldsymbol{A}\cdot\boldsymbol{B} = \boldsymbol{B}\cdot\boldsymbol{A} = AB\cos\theta = (5)(8)\cos(30^\circ) = 34.6.$$

Cross product of vectors: The second kind of vector product is called the *cross product* or the *vector product*. It differs from the scalar product in that it produces a new *vector*, not a scalar like the scalar product. The cross product is defined by

$$\boldsymbol{A}\times\boldsymbol{B} = (AB\sin\theta)\hat{\boldsymbol{I}}, \tag{4}$$

where θ is again the angle between the vectors and $\hat{\boldsymbol{I}}$ is a unit vector that is perpendicular to the plane containing the vectors $\boldsymbol{A}$ and $\boldsymbol{B}$, with its direction (up or down) given by the right-hand rule:



Rotate ***A*** into ***B*** with the right hand. The thumb points in the direction (up or down) of the new vector ***A*** x ***B***.

Unlike for the scalar product, the order in the cross product matters:

$$\boldsymbol{A} \times \boldsymbol{B} = -\boldsymbol{B} \times \boldsymbol{A}$$

(easily seen from the right-hand rule: try to rotate $\boldsymbol{B}$ into $\boldsymbol{A}$ in the preceding diagram and note the direction that your thumb points).

Note some special cases of the vector product:

1. If $\boldsymbol{A}$ and $\boldsymbol{B}$ point in the same or opposite directions, $\sin\theta = 0$ and $|\boldsymbol{A} \times \boldsymbol{B}| = 0$.
2. If $\boldsymbol{A}$ and $\boldsymbol{B}$ are perpendicular, $\sin\theta = 1$ and $|\boldsymbol{A} \times \boldsymbol{B}| = AB$.

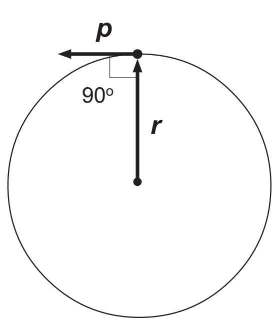
Example: Angular Momentum

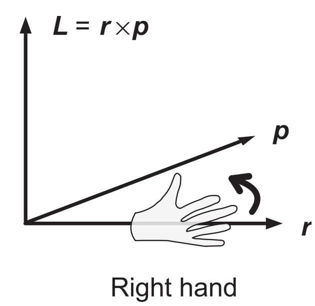
Angular momentum $\boldsymbol{L}$ is a vector that measures the tendency of a body in angular motion to remain in that motion. It is conserved (the reason an ice skater spins faster if the arms are drawn in is conservation of angular momentum).

Angular momentum with respect to some coordinate system is the cross product of the position vector with the momentum vector:

$$\boldsymbol{L} \equiv \boldsymbol{r} \times \boldsymbol{p} = \boldsymbol{r} \times (m\boldsymbol{v}). \tag{5}$$

For example, consider the angular momentum associated with uniform circular motion





The magnitude of the angular momentum is

$$L = rp \sin 90^\circ = rp$$

and the direction is out of the paper, as illustrated by the right-hand rule in the figure. The SI units for angular momentum are $\mathrm{kg\,m^2 s^{-1}}$.

3.4 Conservation Laws

The case of angular momentum just considered is an example of a quantity that is *conserved* by all interactions in Newtonian physics. Conservation of angular momentum is an example of a *conservation law*. In Newtonian physics, we believe that

- Energy
- Momentum
- Mass
- Angular momentum

are always conserved in isolated systems. Conservation laws are very important. Since they *must be obeyed, no matter what,* they often can be used to simplify the solution of problems. We will see specific examples shortly.

3.5 Newton's Three Laws of Motion

Newton's 1st Law:

Objects in a state of uniform motion remain in that state of motion unless an external force acts on them (*The law of inertia)*.

Newton's 2nd Law:

If an external force $\boldsymbol{F}$ acts on an object, the acceleration $\boldsymbol{a}$ experienced by the object is given by the force divided by the mass, so that:

$$\boldsymbol{F} = m\boldsymbol{a}$$

This permits the change in velocity (acceleration) to be computed.

The force $\boldsymbol{F}$ in this case is the vector sum of all forces acting on the object:

$$\boldsymbol{F} = \boldsymbol{F}_1 + \boldsymbol{F}_2 + \ldots + \boldsymbol{F}_n = \sum_{i=1}^{n} \boldsymbol{F}_i .$$

Assuming constant mass, Newton's 2nd law may be written in the equivalent forms

$$\boldsymbol{F} = m\boldsymbol{a} = m\frac{d\boldsymbol{v}}{dt} = \frac{d(m\boldsymbol{v})}{dt} = \frac{d\boldsymbol{p}}{dt}. \tag{6}$$

These vector equations are equivalent to three simultaneous equations in the components (in 3-D). For example

$$\boldsymbol{F} = m\boldsymbol{a} \rightarrow \begin{cases} F_x = ma_x \\ F_y = ma_y \\ F_z = ma_z . \end{cases} \tag{7}$$

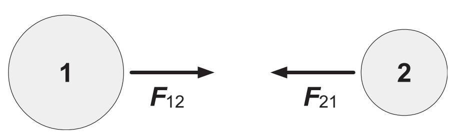
The standard unit of force in the SI system is the Newton:

$$1 \text{ Newton} \equiv 1 \text{ N} = 1 \text{ kg m s}^{-2}.$$

Newton's 3rd Law:

For every reaction, there is an equal and opposite reaction.

Notice that in Newton's 3rd law the action and reaction are forces that always act on *different objects* (never on the same object):



If object 1 exerts a force $\boldsymbol{F}_{21}$ on object 2, then object 2 exerts a force $\boldsymbol{F}_{12} = -F_{21}$ on object 1. These forces are equal in magnitude but opposite in direction.

3.6 Newton's Universal Law of Gravitation

Newton reasoned that gravity was a force, obeying his three laws of motion.

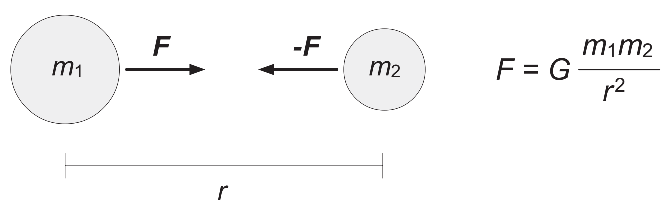
3.6.1 The gravitational force

From the observed properties of gravity, Newton deduced his *Universal Law of Gravitation:*

Universal Law of Gravitation

Every mass in the Universe exerts a force on every other mass that is attractive and directed along the line of centers for the two masses, with the magnitude of the force given by

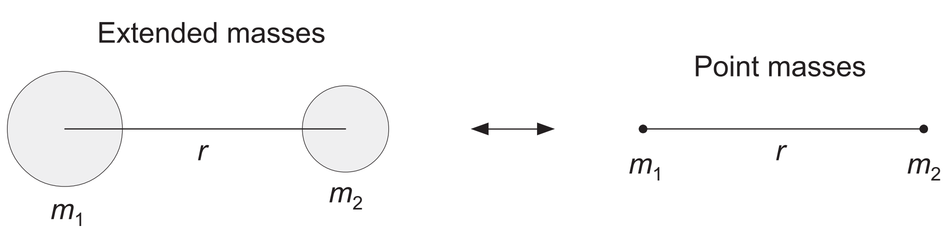
$$F \equiv |\boldsymbol{F}| = G\frac{m_1 m_2}{r^2}, \tag{8}$$



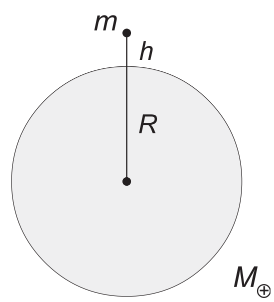
where the *universal gravitational constant* G is measured to be

$$\begin{aligned} G &= 6.673 \times 10^{-11}\ \mathrm{N\,m^2kg^{-2}} \\ &= 6.673 \times 10^{-11}\ \mathrm{kg^{-1}m^3s^{-2}}. \end{aligned} \tag{9}$$

An important property of the gravitational force is that we can prove (see Carrol and Ostlie) that for a spherical mass distribution exerting a gravitational force on a point mass outside the mass distribution, the gravitational force is exactly as if all the mass of the mass distribution were concentrated at its center.



For an object of mass m at the surface of the Earth or a height h above it



the magnitude of the gravitational force acting on m is

$$F = G\frac{M_\oplus m}{(R+h)^2}.$$

But by Newton's 2nd law $F = ma$, so by comparing the *local acceleration due to gravity* is given by

$$g \equiv G\frac{M_\oplus}{(R+h)^2},$$

and we can write

$$F = mg$$

Typically, near the surface of the Earth ($h \sim 0$) we measure that

$$g \simeq 9.8\ \mathrm{m\,s^{-2}}.$$

Let us check this explicitly:

$$M_\oplus = 5.97 \times 10^{24}\ \mathrm{kg} \qquad R_\oplus = 6.38 \times 10^6\ \mathrm{m}.$$

Therefore, the gravitational acceleration should be

$$\begin{aligned} g &= G\frac{M_\oplus}{R^2} \\ &= 6.673 \times 10^{-11}\ \mathrm{kg^{-1} m^3 s^{-2}} \left(\frac{5.97 \times 10^{24}\ \mathrm{kg}}{(6.38 \times 10^6\ \mathrm{m})^2}\right) \\ &= 9.79\ \mathrm{m\,s^{-2}}. \end{aligned}$$

Example: Acceleration Due to Earth's Gravity at the Moon's Orbit

What is the the local gravitational acceleration due to Earth at a distance equal to the Moon's orbit (ignoring the gravity of the Moon)? The Moon's orbit is about 384,000 km from the center of the Earth, so

$$\begin{aligned} g &= G\frac{M_\oplus}{(R+h)^2} \\ &= 6.673 \times 10^{-11}\ \mathrm{kg^{-1}m^3s^{-2}} \left(\frac{5.97\times 10^{24}\ \mathrm{kg}}{(3.84\times 10^{8}\ \mathrm{m})^2}\right) \\ &= 0.0027\ \mathrm{m\,s^{-2}}. \end{aligned}$$

Thus, the ratio of the gravitational force exerted by the Earth on a mass at its surface to the force exerted on that same mass at the distance of the Moon is

$$\frac{F_1}{F_2} = \frac{mg_1}{mg_2} = \frac{g_1}{g_2} = \frac{9.79}{0.0027} \simeq 3626.$$

3.6.2 Weight and mass

Weight and mass are not the same thing. Weight is the gravitational force exerted on a mass,

$$\text{Weight} = \text{Force} = mg.$$

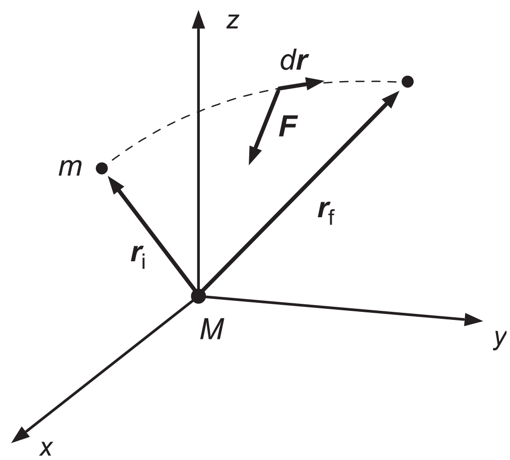
Its SI units are Newtons (N). In the English system the unit of weight is the pound (lb), with the conversion 1 lb = 4.448 N.

The mass of an object is constant but its weight depends on its location (because it depends on the local gravitational acceleration).

3.6.3 Gravitational potential energy

Energy is conserved in physical processes. The total energy of an object is generally a sum of a kinetic energy (energy of motion) and a potential energy. If an object is in a gravitational field, its gravitational potential energy can change if it changes its location.

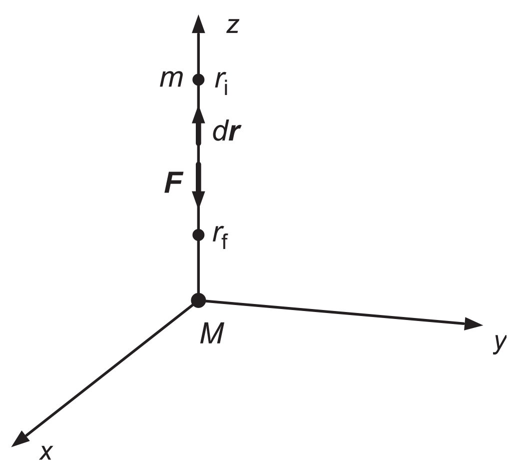
Consider a mass m that moves from a position $\boldsymbol{r}_{\mathrm{i}}$ to $\boldsymbol{r}_{\mathrm{f}}$ in a gravitational field that is generated by a mass M at the origin of the coordinate system:



The gravitational force $\boldsymbol{F}$ exerted on m is directed toward the origin and the definition of the change in potential energy is

$$\Delta U = U_{\mathrm{f}} - U_{\mathrm{i}} = -\int_{\boldsymbol{r}_{\mathrm{i}}}^{\boldsymbol{r}_{\mathrm{f}}} \boldsymbol{F} \cdot d\boldsymbol{r}.$$

Let's take a simple case of a mass m moving vertically along the z axis.



Then the scalar product is easy since the position vectors and the force vector are pointed in opposite directions ($\theta = \pi$) and

$$\begin{aligned}\Delta U &= -\int_{\boldsymbol{r}_{\mathrm{i}}}^{\boldsymbol{r}_{\mathrm{f}}} \boldsymbol{F}\cdot d\boldsymbol{r} = \int_{\boldsymbol{r}_{\mathrm{i}}}^{\boldsymbol{r}_{\mathrm{f}}} F\cos(\pi)dr = -\int_{\boldsymbol{r}_{\mathrm{i}}}^{\boldsymbol{r}_{\mathrm{f}}} -F dr \\ &= \int_{\boldsymbol{r}_{\mathrm{i}}}^{\boldsymbol{r}_{\mathrm{f}}} G\frac{Mm}{r^2}dr = -\frac{GMm}{r}\bigg|_{r_{\mathrm{i}}}^{r_{\mathrm{f}}} \\ &= -GMm\left(\frac{1}{r_{\mathrm{f}}} - \frac{1}{r_{\mathrm{i}}}.\right).\end{aligned}$$

This is the change in the gravitational potential energy. We make three general remarks about it

1. A more general derivation would have shown that the result is *independent of path*, depending only on the endpoints $\boldsymbol{r}_{\mathrm{i}}$ and $\boldsymbol{r}_{\mathrm{f}}$.
2. Generally, only changes in the potential are relevant and we can define an arbitrary zero for the gravitational potential energy scale. It is conventional to choose

   $$U \to 0 \quad \text{as} \qquad r_{\mathrm{i}} \to \infty.$$

   Then $1/r_{\mathrm{i}} \to 0$ and (dropping subscripts) we may write for the *gravitational potential*

   $$U \equiv -\frac{GMm}{r}. \tag{10}$$

   Conventional: we could choose any zero for the scale if we wished.
3. The magnitude of the gravitational force is obtained from the derivative of the gravitational potential,

   $$F_{\mathrm{grav}} = -\frac{dU}{dr} = -\frac{d}{dr}\left(\frac{-GMm}{r}\right) = -\frac{GMm}{r^2},$$

   where the minus sign indicates that it is attractive.

3.6.4 Escape velocity

The escape velocity from a gravitational field is a useful concept:

Escape Velocity

The initial vertical component of velocity from a given location that gives $v \to 0$ as $r \to \infty$.

We may derive a formula for escape velocity simply by using conservation of energy. The total energy of some mass m moving in a gravitational field is

$$E = \frac{1}{2}mv^2 - G\frac{Mm}{r} = E_{\text{kinetic}} + E_{\text{potential}}.$$

As $r \to \infty$, by definition $v \to 0$, so at infinity,

$$E_{\text{kinetic}} \to 0 \qquad E_{\text{potential}} \to \infty \qquad E \to 0.$$

Thus, at infinity the total energy is zero. But since energy is conserved, the total energy must be zero at any r. Setting $E = 0$ in the preceding equation gives

$$\tfrac{1}{2}mv^2 - G\frac{Mm}{r} = 0,$$

which may be solved for v to give

$$v_{\text{esc}} = \sqrt{\frac{2GM}{r}}. \tag{11}$$

Notice that

- The mass of the object m has cancelled out. The escape velocity depends only on the properties of the gravitational field, not on the mass of the object that is escaping.
- The escape velocity depends on where we start from (r in the preceding formula). The escape velocity from the surface of the Earth is greater than the escape velocity from an orbit 200 km above the surface of the Earth, for example.

Example: Escape velocity from Earth's surface

$$\begin{aligned} v_{\rm esc} &= \sqrt{\frac{2GM}{r}} = \sqrt{\frac{2GMr}{r^2}} = \sqrt{2gr} \\ &= \sqrt{2(9.8\ \mathrm{m\,s^{-2}})(6.38\times 10^6\ \mathrm{m})} \\ &= 11,182\ \mathrm{m\,s^{-1}} = 11.2\ \mathrm{km\,s^{-1}}. \end{aligned}$$

Example: Escape velocity from Jupiter's surface

$$\begin{aligned} v_{\rm esc} &= \sqrt{\frac{2GM}{r}} \\ &= \sqrt{\frac{2(6.673\times 10^{-11}\ \ \mathrm{kg^{-1}m^3s^{-2}}\ )(1.9\times 10^{27}\ \mathrm{kg})}{7.149\times 10^7\ \mathrm{m}}} \\ &= 59,556\ \mathrm{m\,s^{-1}} = 59.6\ \mathrm{km\,s^{-1}}. \end{aligned}$$

Example: Escape velocity from Sun's surface

$$\begin{aligned} v_{\rm esc} &= \sqrt{\frac{2GM}{r}} \\ &= \sqrt{\frac{2(6.673\times 10^{-11}\ \ \mathrm{kg^{-1}m^3s^{-2}}\ )(1.99\times 10^{30}\ \mathrm{kg})}{6.96\times 10^8\ \mathrm{m}}} \\ &= 617,728\ \mathrm{m\,s^{-1}} = 618\ \mathrm{km\,s^{-1}}. \end{aligned}$$

Example: Escape velocity from surface of Phobos

The Martian moon Phobos is not spherical but its average radius is about 11 km. Using this and its mass of 1.08×10^{16} kg, we may estimate that

$$\begin{aligned} v_{\rm esc} &= \sqrt{\frac{2GM}{r}} \\ &= \sqrt{\frac{2(6.673\times 10^{-11}\ \ \mathrm{kg^{-1}m^3s^{-2}}\ )(1.08\times 10^{16}\ \mathrm{kg})}{11\times 10^3\ \mathrm{m}}} \\ &= 11.4\ \mathrm{m\,s^{-1}}. \end{aligned}$$

That's not very much. Could a basketball player with Michael Jordan leaping ability attain escape velocity on Phobos just by jumping straight up?

Suppose a basketball player can leap vertically by 40 inches (about one meter) on Earth. By energy conservation again we have

$$\tfrac{1}{2}mv_1^2 + mgy_1 = \tfrac{1}{2}mv_2^2 + mgy_2,$$

where quantities on the left side refer to the player on the floor, and the quantities on the right side to the player at the top of his leap. Dividing through by m and rearranging,

$$v_1^2 - v_2^2 = 2g(y_2 - y_1).$$

But v_2 is at the top of the leap so it is equal to zero, and $y_2 - y_1 = \Delta y$ is just the vertical leap of 1 meter. Therefore, solving for the initial velocity v_1,

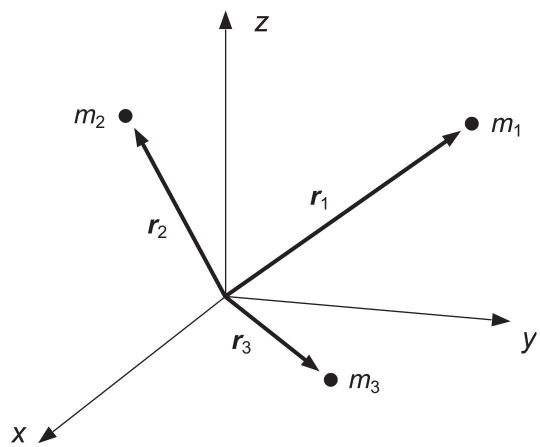
$$v_1 = \sqrt{2g\Delta y} = \sqrt{2(9.8\ \mathrm{m\,s^{-2}})(1\ \mathrm{m})} = 4.4\ \mathrm{m\,s^{-1}}.$$

So Michael Jordan could not launch into orbit by jumping from Phobos, but he wouldn't miss it by very far!

For further reference, a world-class sprinter can attain a speed of a little over 10 $\mathrm{m\,s^{-1}}$, UT softball pitcher Monica Abbot's 70 mph fastball corresponds to about 31 $\mathrm{m\,s^{-1}}$, and a hard kick in a world cup football (soccer) match, or the serve of a top tennis player, can reach initial speeds in the vicinity of 50–60 $\mathrm{m\,s^{-1}}$. (Convenient conversion: 1 mph is 0.447 $\mathrm{m\,s^{-1}}$.)

3.6.5 Center of mass reference frame

Consider a collection of masses m_i at position coordinates $\boldsymbol{r}_i$,



Define a position vector $\boldsymbol{R}$ that is the weighted average

$$\boldsymbol{R} = \frac{\sum_{i=1}^{n} m_i \boldsymbol{r}_i}{\sum_{i=1}^{n} m_i}.$$

The position $\boldsymbol{R}$ is termed the *center of mass*. The total mass is $M = \sum_{i=1}^{n} m_i$ so

$$M\boldsymbol{R} = \sum_{i=1}^{n} m_i \boldsymbol{r}_i.$$

Assume the masses to be constant and differentiate

$$M\frac{d\boldsymbol{R}}{dt} = \sum_{i=1}^{n} m_i \frac{d\boldsymbol{r}_i}{dt}$$

which is equivalent to

$$M\boldsymbol{V} = \sum_{i=1}^{n} m_i \boldsymbol{v}_i$$

and also to

$$\boldsymbol{P} = \sum_{i=1}^{n} \boldsymbol{p}_i,$$

where $\boldsymbol{V}$ in the CM velocity and $\boldsymbol{P}$ is the CM momentum. Thus, the system behaves as if all mass were concentrated at the CM $\boldsymbol{R}$, moving with the CM velocity $\boldsymbol{V}$ and CM momentum $\boldsymbol{P}$.

Differentiate with respect to t,

$$\frac{d\boldsymbol{P}}{dt} = \sum_{i=1}^{n} \frac{d\boldsymbol{p}_i}{dt}.$$

But if no external forces act on the masses (all forces are internal between the masses) the *total force* must be zero, by Newton's third law applied to any two interacting pairs (equal and opposite forces for each pair). Therefore

$$\boldsymbol{F} = \frac{d\boldsymbol{P}}{dt} = M\frac{d^2\boldsymbol{R}}{dt^2} = 0.$$

> *The center of mass does not accelerate if there are no external forces acting on the system of masses.*

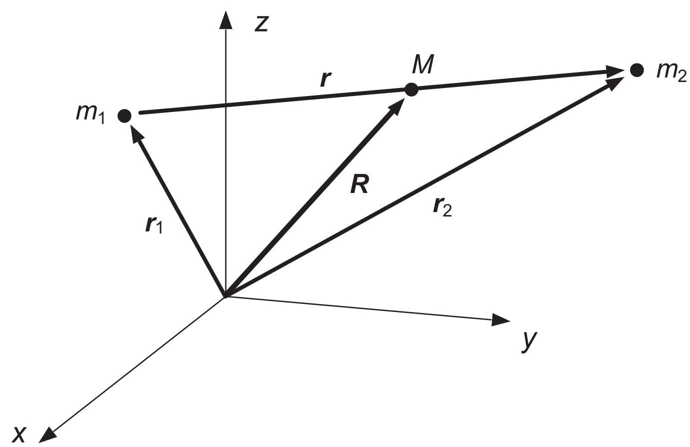
This implies that we may simplify the manybody problem by choosing a coordinate system for which

$$\boldsymbol{R} = 0 \qquad \boldsymbol{V} = 0.$$

This is called the CM frame. It is an *inertial frame* (one in which Newton's first law is valid).

3.6.6 Center of mass for a binary system

Consider the important special case of two masses (binary system):



$$\boldsymbol{r}_1 + \boldsymbol{r} = \boldsymbol{r}_2 \quad \rightarrow \quad \boldsymbol{r} = \boldsymbol{r}_2 - \boldsymbol{r}_1.$$

Then choosing the center of mass as the origin,

$$\boldsymbol{R} = \frac{m_1 \boldsymbol{r}_1 + m_2 \boldsymbol{r}_2}{m_1 + m_2} = 0$$

Therefore,

$$m_1 \boldsymbol{r}_1 + m_2 \boldsymbol{r}_2 = 0,$$

and since $\boldsymbol{r}_2 = \boldsymbol{r}_1 + \boldsymbol{r}$,

$$\begin{aligned} m_1 \boldsymbol{r}_1 + m_2(\boldsymbol{r}_1 + \boldsymbol{r}) &= 0 \\ m_1 \boldsymbol{r}_1 + m_2 \boldsymbol{r}_1 &= -m_2 \boldsymbol{r} \\ \boldsymbol{r}_1 = -\frac{m_2}{m_1 + m_2} \boldsymbol{r} &= -\frac{m_2}{M} \boldsymbol{r}. \end{aligned}$$

By a similar proof,

$$\boldsymbol{r}_2 = \frac{m_1}{m_1 + m_2} \boldsymbol{r} = \frac{m_1}{M} \boldsymbol{r}.$$

Introducing the *reduced mass* μ

$$\mu \equiv \frac{m_1 m_2}{m_1 + m_2},$$

we can write

$$\begin{aligned} \boldsymbol{r}_1 &= -\frac{m_2}{m_1 + m_2} \boldsymbol{r} = -\frac{m_2 m_1}{m_1(m_1 + m_2)} \boldsymbol{r} \\ &= -\frac{\mu}{m_1} \boldsymbol{r}. \end{aligned}$$

By a similar proof,

$$\boldsymbol{r}_2 = \frac{\mu}{m_2}\boldsymbol{r}.$$

Total energy: The utility of the CM system can be seen in writing the total energy of the binary system,

$$E = \tfrac{1}{2}m_1|\boldsymbol{v}_1|^2 + \tfrac{1}{2}m_2|\boldsymbol{v}_2|^2 - G\frac{m_1 m_2}{|\boldsymbol{r}_2 - \boldsymbol{r}_1|}.$$

Substituting and rearranging,

$$E = \tfrac{1}{2}\mu v^2 - G\frac{M\mu}{r},$$

where $r = |\boldsymbol{r}| = |\boldsymbol{r}_2 - \boldsymbol{r}_1|$ and $v = |\boldsymbol{v}|$, with

$$\boldsymbol{v} = \frac{d\boldsymbol{r}}{dt} = \frac{d}{dt}(\boldsymbol{r}_2 - \boldsymbol{r}_1) = \boldsymbol{v}_2 - \boldsymbol{v}_1.$$

The total energy is now the sum of the kinetic energy of the reduced mass μ and the potential energy of the reduced mass moving about the total mass $M = m_1 + m_2$ at the origin.

Orbital angular momentum: The orbital angular momentum for the binary is

$$\begin{aligned}\boldsymbol{L} &= \boldsymbol{r}_1 \times \boldsymbol{p}_1 + \boldsymbol{r}_2 \times \boldsymbol{p}_2 \\ &= m_1\boldsymbol{r}_1 \times \boldsymbol{v}_1 + m_2\boldsymbol{r}_2 \times \boldsymbol{v}_2.\end{aligned}$$

Substituting gives

$$\begin{aligned}\boldsymbol{L} &= m_1\left(-\frac{\mu}{m_1}\right)\boldsymbol{r} \times \boldsymbol{v}_1 + m_2\left(\frac{\mu}{m_2}\right)\boldsymbol{r} \times \boldsymbol{v}_2 \\ &= -\mu\boldsymbol{r} \times \boldsymbol{v}_1 + \mu\boldsymbol{r} \times \boldsymbol{v}_2 \\ &= \mu\boldsymbol{r} \times (\boldsymbol{v}_2 - \boldsymbol{v}_1) \\ &= \mu\boldsymbol{r} \times \boldsymbol{v},\end{aligned}$$

so the total $\boldsymbol{L}$ is the angular momentum of the *reduced mass only.*

The binary problem of calculating the motion of two bodies has been replaced by the calculation of the motion of a single effective mass (the reduced mass μ) about a stationary point containing the total mass M of the system, with the separation between M and μ given by the separation between m_1 and m_2.

Binary center of mass: Choosing the CM as the origin for the binary,

Take the line of centers as the x axis. From the preceding equations,

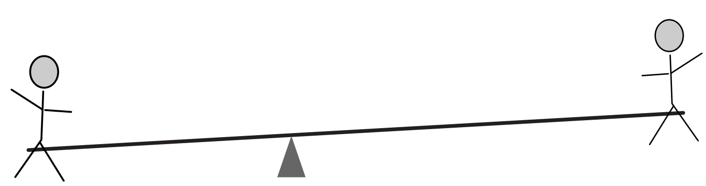
$$\frac{\boldsymbol{r}_1}{\boldsymbol{r}_2} = \frac{-(\mu/m_1)\boldsymbol{r}}{(\mu/m_2)\boldsymbol{r}} = -\frac{m_2}{m_1} = \frac{\boldsymbol{r}_1^x}{\boldsymbol{r}_2^x}.$$

But from the diagram

$$\boldsymbol{r}_1^x = -|\boldsymbol{r}_1| = -r_1 \qquad \boldsymbol{r}_2^x = |\boldsymbol{r}_2| = r_2$$

Substituting these gives the *seesaw equation*,

$$r_1 m_1 = r_2 m_2 \qquad \text{where } r_1 + r_2 = r = \text{distance between masses.}$$



Two special cases are of interest.

- Suppose that $m_1 = m_2$. Then

$$r_1 m_1 = r_2 m_2 \quad \longrightarrow \quad r_2 = r_1 = \tfrac{1}{2} r,$$

and the CM lies halfway between the masses.

- Suppose one mass much larger than the other, $m_1 >> m_2$. Then

$$\frac{r_1}{r_2} = \frac{m_2}{m_1} \simeq 0.$$

Therefore, the CM almost coincides with the center of the large mass.

Example: Center of mass for the Earth–Sun system

We have

$$M_\odot = 1.99 \times 10^{30}\ \text{kg} \qquad M_\oplus = 5.97 \times 10^{24}\ \text{kg} \qquad r = 1.496 \times 10^{11}\ \text{m}$$

and we must solve simultaneously

$$\frac{r_2}{r_1} = \frac{M_\oplus}{M_\odot} \qquad r_1 = r - r_2.$$

Substituting the right equation into the left,

$$\frac{r_2}{r - r_2} = \frac{M_\oplus}{M_\odot},$$

which may be solved for r_2 to give

$$r_2 = \frac{M_\oplus / M_\odot}{1 - M_\oplus / M_\odot}\, r.$$

For the Earth–Sun system,

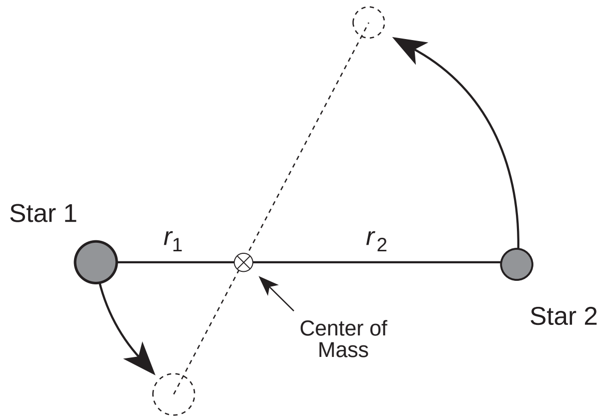
$$\frac{M_\oplus}{M_\odot} = \frac{5.97 \times 10^{24}\ \text{kg}}{1.99 \times 10^{30}\ \text{kg}} = 3 \times 10^{-6}.$$

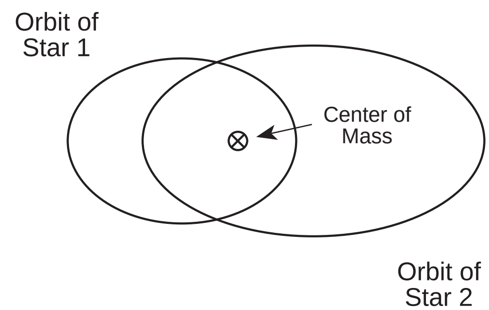
Neglecting this term in the denominator,

$$r_2 \simeq \frac{M_\oplus}{M_\odot}\, r = (3 \times 10^{-6})(1.496 \times 10^{11}\ \text{m}) \simeq 4.5 \times 10^5\ \text{m}.$$

For reference, $R_\odot \simeq 7 \times 10^8$ m, so the CM of the Earth–Sun system is well inside the Sun.

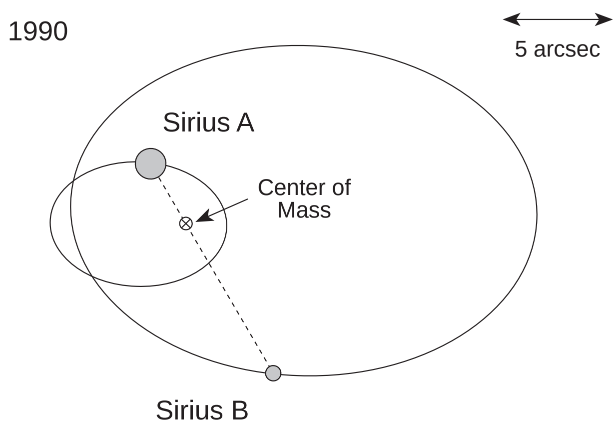
Elliptical Kepler motion in the CM system for a binary star:





(See the Java applet OJTA 4.25 for binary motion.)

Actual example of ellipical motion for the Sirius B system:



3.7 Kepler's Laws from Newton's Law of Gravitation

We now outline how Kepler's laws follow from Newton's law of gravitation

3.7.1 Kepler's first law

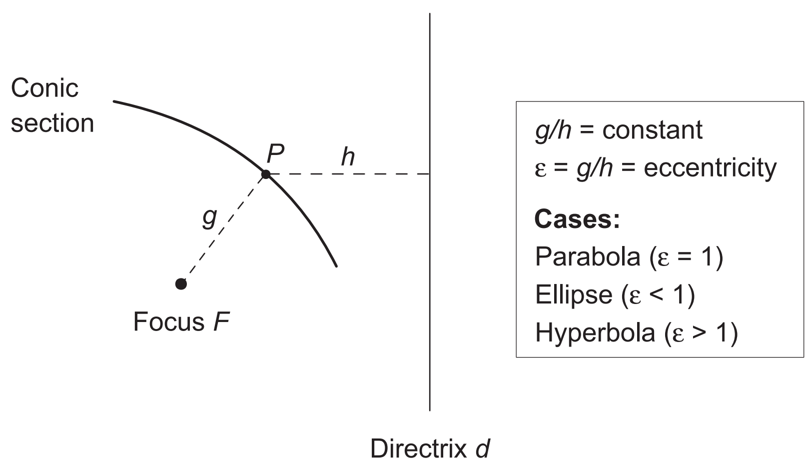
By considering the effect of gravity on the orbit of a planet in the CM sysem (see Ostlie and Carrol, pp. 43–45), we can prove

$$r = \frac{L^2/\mu^2}{GM(1+\epsilon\cos\theta)}. \tag{12}$$

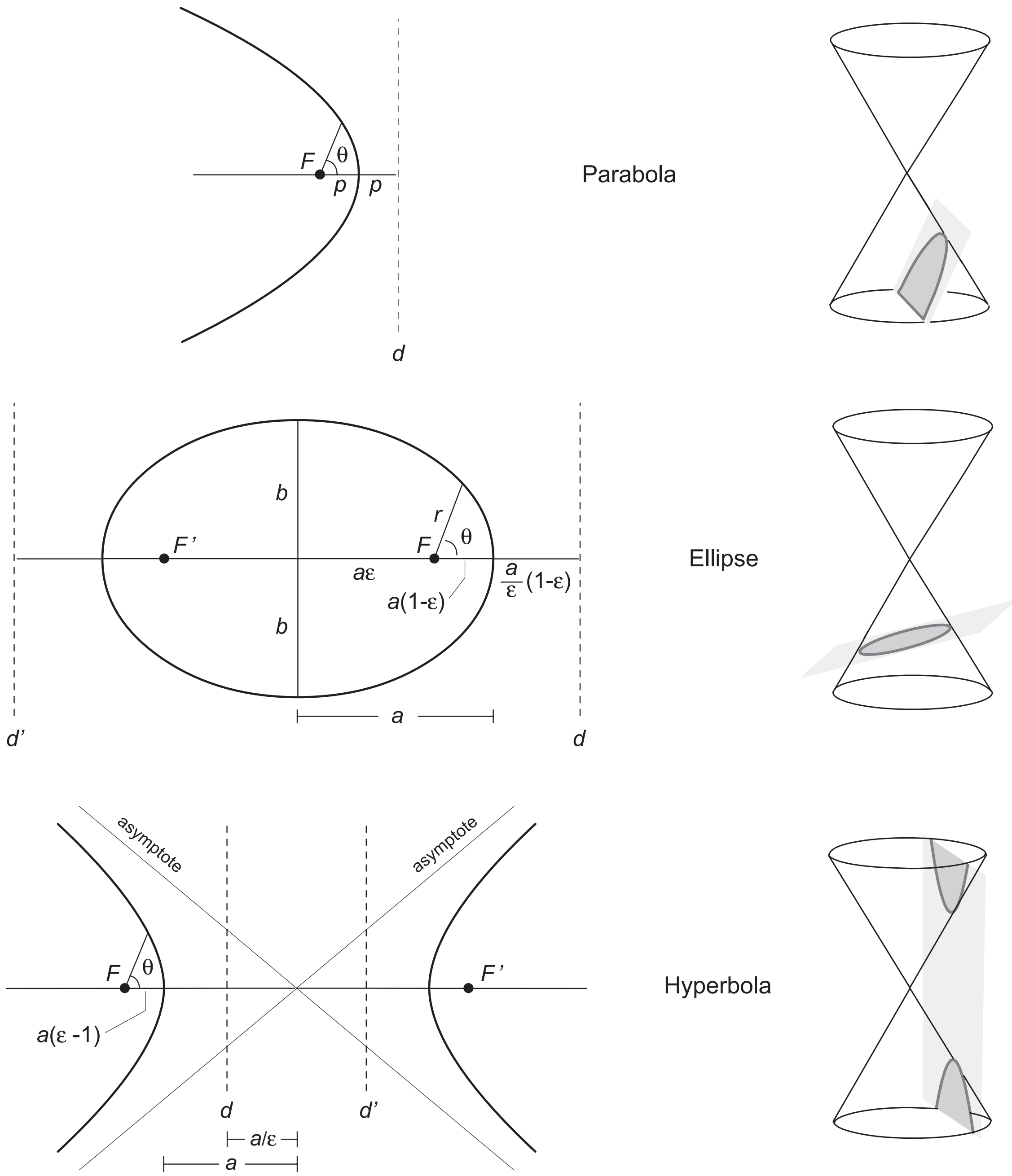
But this is the equation of a *conic section*, which corresponds to equations for parabolas, ellipses, and hyperbolas:

$$\begin{aligned} r &= \frac{L^2/\mu^2}{GM(1+\epsilon\cos\theta)} \\ &= \begin{cases} \dfrac{2p}{1+\epsilon\cos\theta} & \left(\text{parabola}, \quad \epsilon = 1,\ \ 2p = \dfrac{L^2/\mu^2}{GM}\right) \\ \dfrac{a(1-\epsilon^2)}{1+\epsilon\cos\theta} & \left(\text{ellipse}, \quad \epsilon < 1,\ \ a(1-\epsilon^2) = \dfrac{L^2/\mu^2}{GM}\right) \\ \dfrac{a(\epsilon^2-1)}{1+\epsilon\cos\theta} & \left(\text{hyperbola}, \quad \epsilon > 1,\ \ a(\epsilon^2-1) = \dfrac{L^2/\mu^2}{GM}\right) \end{cases} \end{aligned} \tag{13}$$

The geometrical definition of a conic section is summarized in the following figure

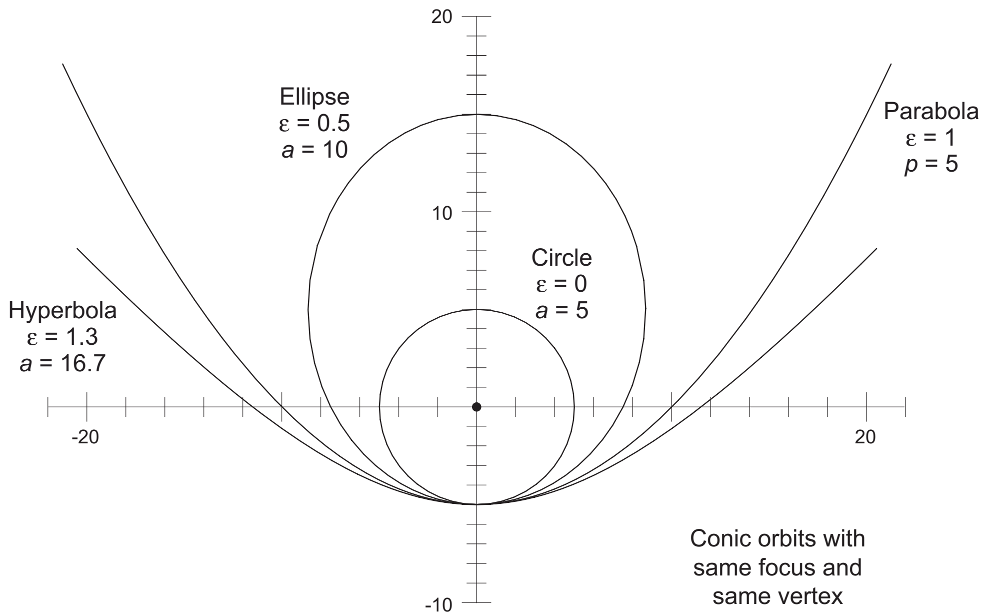


The three general types of conic sections are parabolas, ellipses, and hyperbolas (the circle is a special case of an ellipse). The geometrical properties of parabolas, ellipses, and hyperbolas are summarized in the following figure.



So “orbits of the planets are ellipses” generalizes to “orbits in gravitational fields are conic sections”, with the ellipse as a special case for a bound orbit.

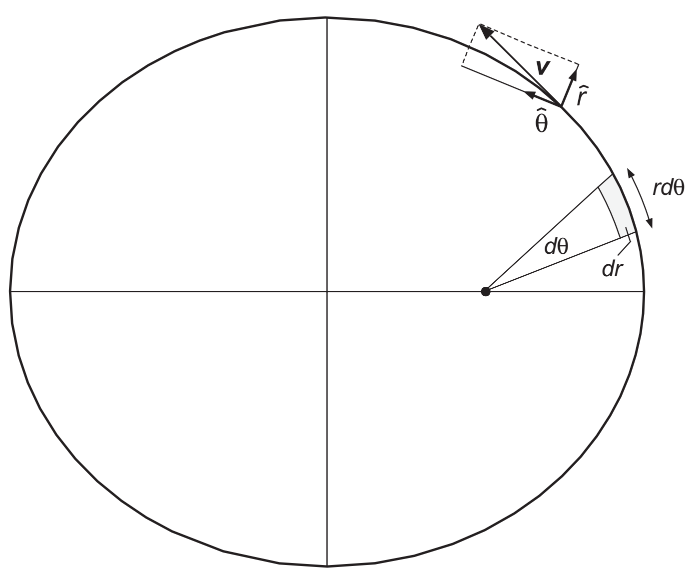
Examples of conic-section gravitational orbits are shown in the following figure,



which shows conic section orbits having the same focus and the same vertex (distance of closest approach to the focus).

3.7.2 Kepler's second law

Consider the following ellipse:



The differential area of the shaded strip is $rd\theta$, so the differential area swept out by the angle $d\theta$ is

$$dA = d\theta \int_0^r r dr = d\theta(\tfrac{1}{2}r^2)\Big|_0^r = \tfrac{1}{2}r^2 d\theta$$

and the rate of change of the area swept out in a time dt is

$$\frac{dA}{dt} = \tfrac{1}{2}r^2\frac{d\theta}{dt}.$$

From the diagram, the velocity vector $\boldsymbol{v}$ can be resolved into components v_r along the radial drawn from the focus and v_θ perpendicular to the radial. In terms of unit vectors $\hat{r}$ and $\hat{\theta}$ in these directions,

$$\boldsymbol{v} = v_r\hat{r} + v_\theta\hat{\theta} = \frac{dr}{dt}\hat{r} + r\frac{d\theta}{dt}\hat{\theta}$$

Therefore, $v_\theta = rd\theta/dt$ and substituting $d\theta/dt = v_\theta/r$ into the earlier equation gives

$$\frac{dA}{dt} = \tfrac{1}{2}r^2\frac{v_\theta}{r} = \tfrac{1}{2}rv_\theta.$$

Since $\hat{r}$ and $\hat{\theta}$ are orthogonal, $|\boldsymbol{r}\times\boldsymbol{v}| = rv_\theta \sin(90°) = rv_\theta$ and

$$\begin{aligned} rv_\theta &= |\boldsymbol{r}\times\boldsymbol{v}| = \frac{1}{\mu}|\boldsymbol{r}\times\underbrace{(\mu\boldsymbol{v})}_{\boldsymbol{p}}| \\ &= \frac{1}{\mu}|\underbrace{\boldsymbol{r}\times\boldsymbol{p}}_{\boldsymbol{L}}| = \frac{L}{\mu}, \end{aligned}$$

where L is the magnitude of the orbital angular momentum. Therefore, the change in area is

$$\frac{dA}{dt} = \tfrac{1}{2}rv_\theta = \frac{L}{2\mu} = \text{ constant}$$

since angular momentum and the reduced mass are conserved. But constant dA/dt is just Kepler's 2nd law:

> *The line joining the planet to the focus sweeps out equal areas in equal times.*

3.7.3 Kepler's third law

From the results for Kepler's 2nd law, the total area of the ellipse is

$$\begin{aligned} A &= \oint_{\text{orbit}} dA = \oint_{\text{orbit}} \left(\frac{dA}{dt}\right) dt \\ &= \int_0^P \frac{L}{2\mu} dt = \frac{L}{2\mu} \int_0^P dt = \left.\frac{Lt}{2\mu}\right|_0^P \\ &= \frac{L}{2\mu} P, \end{aligned}$$

where P is the period for one orbit. But we also have from geometry that the area of an ellipse is $A = \pi ab$ [see Eq. (2)], so

$$\pi ab = \frac{L}{2\mu} P,$$

which we can square and solve for P^2 to give.

$$P^2 = \left(\frac{2\mu}{L}\pi ab\right)^2 = \frac{4\mu^2\pi^2a^2b^2}{L^2} = \frac{4\mu^2\pi^2a^2[a^2(1-\epsilon^2)]}{L^2}, \tag{14}$$

where in the last step we have used Eq. (1) for ellipses:

$$b^2 = a^2(1-\epsilon^2). \tag{15}$$

But we can also equate Eqs. (2) and (12)

$$r = \frac{a(1-\epsilon^2)}{1+\epsilon\cos\theta} \qquad r = \frac{L^2/\mu^2}{GM(1-\epsilon\cos\theta)}$$

for ellipses and solve for the angular momentum L to give

$$L = \mu\sqrt{GMa(1-\epsilon^2)}$$

Inserting this into Eq. (14) for P^2 then gives

$$\begin{aligned} P^2 &= \frac{4\mu^2\pi^2a^2[a^2(1-\epsilon^2)]}{\mu^2 GMa(1-\epsilon^2)} \\ &= \left(\frac{4\pi^2}{GM}\right) a^3. \end{aligned}$$

Thus, inserting explicitly that $M = m_1 + m_2$, we arrive at the most general form of Kepler's 3rd law,

$$P^2 = \frac{4\pi^2}{G(m_1 + m_2)} a^3. \tag{16}$$

This expression is valid for any appropriate units and can be written in the form

$$P^2 = ka^3 \qquad k \equiv \frac{4\pi^2}{G(m_1 + m_2)}. \tag{17}$$

Example: Mass of the Earth determined from the Moon's orbit

For the Moon, the period and semimajor axis of the orbit around Earth are

$$P = 27.322 \text{ d} = 2.3606 \times 10^6 \text{ s} \qquad a = 3.844 \times 10^8 \text{ m}.$$

Then from Kepler's 3rd law in the general form (17),

$$M = m_1 + m_2 = \frac{4\pi^2}{G}\frac{a^3}{P^2}$$

Evaluating the constants

$$\frac{4\pi^2}{G} = \frac{4\pi^2}{6.673 \times 10^{-11} \text{ kg}^{-1}\text{m}^3\text{s}^{-2}} = 5.916 \times 10^{11} \text{ kg m}^{-3}\text{s}^2$$

so in these units

$$M = 5.9161 \times 10^{11} \left(\frac{a}{\text{meters}}\right)^3 \left(\frac{\text{seconds}}{P}\right)^2$$

Neglecting the mass of the Moon relative to that of the Earth, $M = M_\oplus + M_{\text{Moon}} \simeq M_\oplus$,

$$\begin{aligned} M_\oplus &= 5.9161 \times 10^{11} \left(\frac{a}{\text{meters}}\right)^3 \left(\frac{\text{seconds}}{P}\right)^2 \\ &= 5.9161 \times 10^{11} \frac{(3.844 \times 10^8)^3}{(2.3606 \times 10^6)^2} \\ &= 6.03 \times 10^{24} \text{ kg}. \end{aligned}$$

If we subtract from this the mass 7.349×10^{22} kg of the Moon, we obtain 5.97×10^{24} kg, which is almost exactly the mass of the Earth.

Let's now demonstrate that we can choose a particular set of units such that numerically $k = 1$, so that we recover Kepler's 3rd law in the form originally proposed by Kepler, if we neglect the mass of the planet relative to the mass of the Sun.

We adopt the following units for time, distance and mass:

$$[\text{time}] = \text{ Earth years (yr)} \qquad [\text{distance}] = \text{ AU} \qquad [\text{mass}] = M_\odot$$

where we use the general notation [X] to denote the units of X. Converting the units of the gravitational constant we can write

$$\begin{aligned}
\frac{4\pi^2}{G} &= \frac{4\pi^2}{6.673 \times 10^{-11}\ \text{kg}^{-1}\text{m}^3\text{s}^{-2}} = 5.916 \times 10^{11}\ \text{kg}\,\text{m}^{-3}\text{s}^2 \\
&= 5.916 \times 10^{11}\ \text{kg}\,\text{m}^{-3}\text{s}^2 \left(\frac{1 M_\odot}{1.989 \times 10^{30}\ \text{kg}}\right) \\
&\qquad\qquad \times \left(\frac{1.496 \times 10^{11}\ \text{m}}{1\ \text{AU}}\right)^3 \left(\frac{1\ \text{yr}}{3.156 \times 10^7\ \text{s}}\right)^2 \\
&= 1 M_\odot\ \text{yr}^2\,\text{AU}^{-3}
\end{aligned}$$

Therefore, employing these units for the factor $\frac{4\pi^2}{G}$, Kepler's 3rd law can be expressed as

$$P^2 = \left(\frac{1 M_\odot}{m_1 + m_2}\right)\left(\frac{a}{\text{AU}}\right)^3\ \text{yr}^2,$$

and if the masses are measured in solar masses and the semimajor axis a in AU, the units of P will be years.

Finally, if we take

$$M = m_\odot + m_{\text{planet}} \simeq m_\odot = 1 M_\odot,$$

the mass factor is just unity and we obtain

$$P^2 = a^3 \qquad \text{(Kepler's 3rd Law)}$$

where a is in AU and P is in Earth years.

If we want to be explicit about the units that are required for Kepler's 3rd law in this form, we could write

$$P^2 = \left(\frac{a^3}{\text{AU}^3}\right)\ \text{yr}^2.$$

for Kepler's 3rd law.