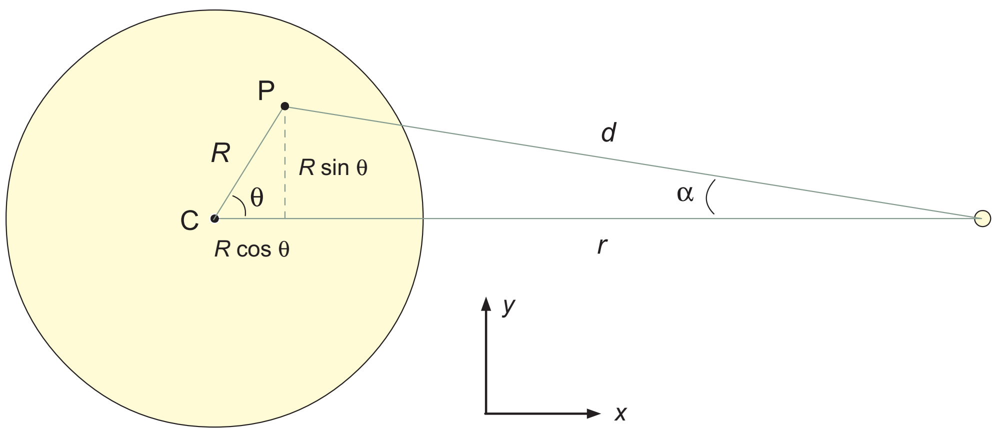
11 Physical Process in the Solar System

Assign: Read Chapter 19 of Carrol and Ostlie (2006)

11.1 Tidal Forces

Two different points inside an extended body feel different gravitational force vectors, as illustrated in the following figure.



This differential gravitational force is the source of *tidal forces*. If we neglect the change in direction, the change in the magnitude of the gravitational force between two nearby points is

$$\frac{\Delta F}{\Delta r} \simeq \frac{dF}{dr} = \frac{d}{dr}\frac{GMm}{r^2} = -2\frac{GMm}{r^3} \quad \rightarrow \quad dF \simeq -2\frac{GMm}{r^3}dr,$$

which tells us a very important thing about tidal forces:

> The gravitational force varies as $1/r^2$ but tidal forces vary as $1/r^3$. Thus, tidal forces depend strongly on the distance between gravitating bodies and go up rapidly as that distance becomes small.

Let's now derive an expression for the tidal force corresponding to the difference in forces experienced by the points C and P in the preceding diagram. Assuming the points to lie in the x–y plane, the components of the the force at C are

$$F_{\mathrm{C}}^{x} = \frac{GMm}{r^2} \qquad F_{\mathrm{C}}^{y} = 0.$$

and the components of the force at P are

$$F_{\mathrm{P}}^{x} = \frac{GMm}{d^2}\cos\alpha \qquad F_{\mathrm{P}}^{y} = -\frac{GMm}{d^2}\sin\alpha.$$

Thus the difference in force between C and P is

$$\begin{aligned}
\Delta\boldsymbol{F} &= \boldsymbol{F}_{\mathrm{P}} - \boldsymbol{F}_{\mathrm{C}} = (F_{\mathrm{P}}^{x} - F_{\mathrm{C}}^{x})\hat{x} + (F_{\mathrm{P}}^{y} - F_{\mathrm{C}}^{y})\hat{y} \\
&= \left(\frac{GMm}{d^2}\cos\alpha - \frac{GMm}{r^2}\right)\hat{x} + \left(-\frac{GMm}{d^2}\sin\alpha - 0\right)\hat{y} \\
&= GMm\left(\frac{\cos\alpha}{d^2} - \frac{1}{r^2}\right)\hat{x} - \frac{GMm}{d^2}\sin\alpha\,\hat{y}.
\end{aligned}$$

But from the right triangle in the diagram and the Pythagorean theorem,

$$\begin{aligned}
d^2 &= (r - R\cos\theta)^2 + (R\sin\theta)^2 \\
&= r^2 - 2rR\cos\theta + R^2\cos^2\theta + R^2\sin^2\theta \\
&= r^2 - 2rR\cos\theta + R^2 \\
&= r^2\left(1 - \frac{2R}{r}\cos\theta + \frac{R^2}{r^2}\right) \\
&\simeq r^2\left(1 - \frac{2R}{r}\cos\theta\right),
\end{aligned}$$

where we have ignored terms of order R^2/r^2. Substituting this into the expression above for $\Delta\boldsymbol{F}$ gives

$$\Delta\boldsymbol{F} = GMm\left(\frac{\cos\alpha}{r^2(1 - 2R\cos\theta/r)} - \frac{1}{r^2}\right)\hat{x} - \frac{GMm\sin\alpha}{r^2(1 - 2R\cos\theta/r)}\hat{y}.$$

Simplify with a binomial expansion assuming $R/r << 1$:

$$\begin{aligned}
\frac{\cos\alpha}{r^2(1 - 2R\cos\theta/r)} &= \frac{\cos\alpha}{r^2}(1 - 2R\cos\theta/r)^{-1} \\
&\simeq \frac{\cos\alpha}{r^2}(1 + 2R\cos\theta/r),
\end{aligned}$$

and likewise,

$$\frac{GMm\sin\alpha}{r^2(1 - 2R\cos\theta/r)} \simeq \frac{GMm\sin\alpha}{r^2}(1 + 2R\cos\theta/r).$$

Substituting these in the expression for $\Delta\boldsymbol{F}$ gives

$$\Delta\boldsymbol{F} \simeq GMm\left(\frac{\cos\alpha(1+2R\cos\theta/r)}{r^2}-\frac{1}{r^2}\right)\hat{x}-\frac{GMm}{r^2}(1+2R\cos\theta/r)\sin\alpha\,\hat{y}.$$

But α is very small so $\cos\alpha \simeq 1$ and from the right triangle

$$\sin\alpha = \frac{R\sin\theta}{d} \simeq \frac{R\sin\theta}{r}$$

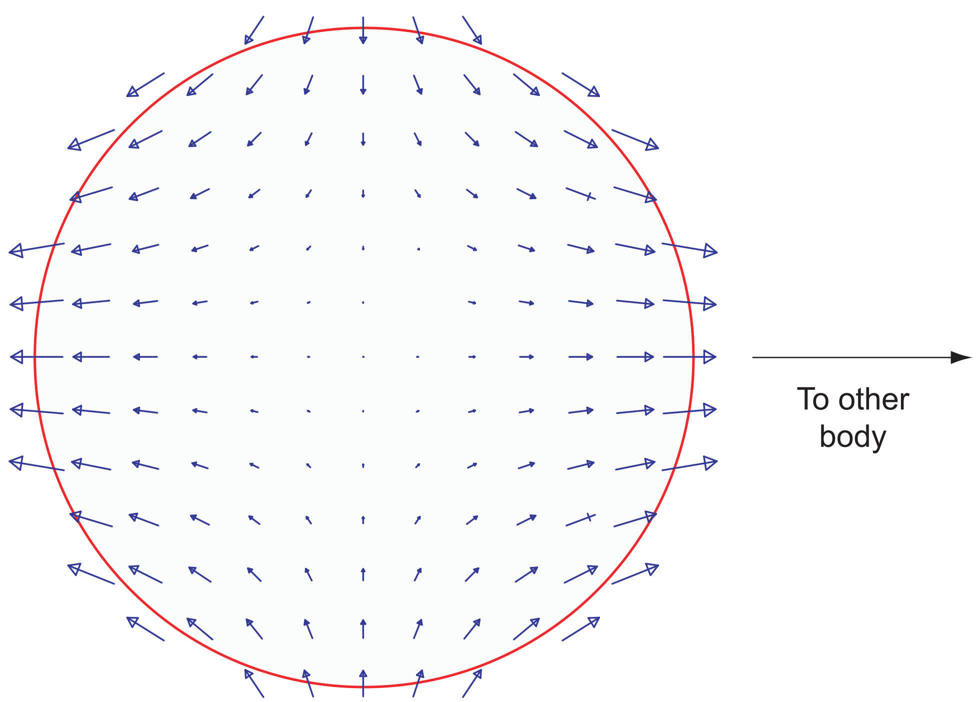
which gives upon substitution

$$\begin{aligned}\Delta\boldsymbol{F} &= \frac{GMm}{r^2}\left(1+\frac{2R}{r}\cos\theta-1\right)\hat{x}-\frac{GMm}{r^2}\left(1+\frac{2R}{r}\cos\theta\right)\frac{R\sin\theta}{r}\hat{y}\\ &= \frac{GMm}{r^2}\left(\frac{2R}{r}\cos\theta\right)\hat{x}-\frac{GMm}{r^2}\left(\frac{R\sin\theta}{r}+\frac{2R^2\cos\theta\sin\theta}{r^2}\right)\hat{y}.\end{aligned}$$

Neglecting the last term because it is of order R^2/r^2, we finally obtain

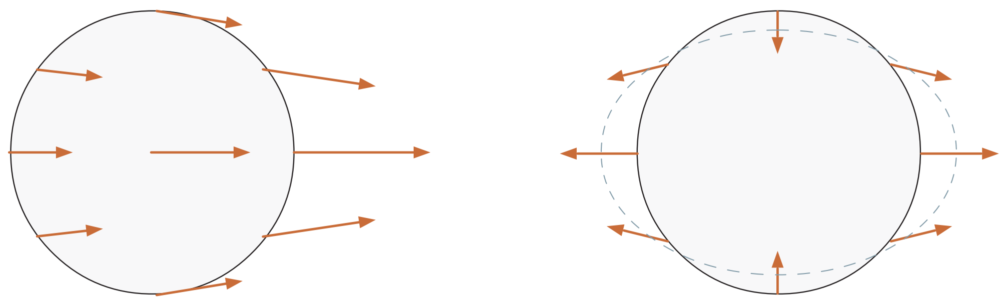
$$\Delta\boldsymbol{F} = \frac{GMmR}{r^3}(2\cos\theta\,\hat{x}-\sin\theta\,\hat{y}). \tag{50}$$

From this formula we deduce the following diagram of tidal force vectors:



Tidal Forces

Thus, the effect of the tidal force is to elongate the object, if it is deformable (the distorted dashed line in the following figure).



This is the origin of the Earth's tides, for example. Notice that in Eq. (50) we have obtained a tidal differential force that depends on $1/r^3$, as expected from the earlier discussion.

11.2 Consequences of Tidal Effects

Tidal effects have a number of important consequences:

1. Tidal forces will distort any deformable body. Thus, not only the oceans, but the body of the Earth is distorted by the lunar gravity. However, because the Earth is rigid compared with the oceans, the tides in the body of the Earth are much smaller than in the oceans.

2. Tidal forces are reciprocal. Not only will the Moon induce tides in the body of the Earth and the Earth's oceans, but by the same argument the gravitational field of the Earth will induce differential forces and therefore tides in the body of the Moon.

3. This reciprocal induction of tides in the body of two objects interacting gravitationally leads to a complicated coupling of the rotational and orbital motions of the two objects that has the following general effects:

   - The tidal coupling of the orbital and rotational motion tends to synchronize them (rotational period $\sim$ orbital period).

- The interior of the Earth and Moon are heated by the tides in their bodies, just as a paper clip is heated by constant bending.

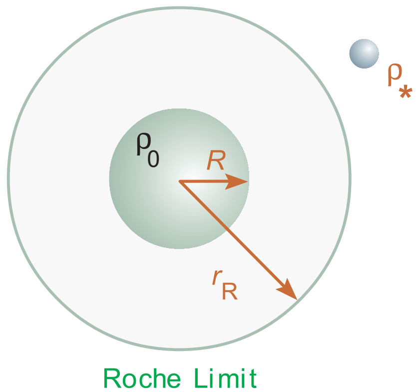
> This effect can be dramatic for objects that experience large tidal forces. For example, the tidal forces exerted by Jupiter on its moon Io are so large that the solid surface of Io is raised and lowered by perhaps hundreds of meters in each rotational period. This motion so heats the interior of Io that it is probably mostly molten; as a consequence, Io is covered with active volcanos and is the geologically youngest object in the Solar System.

- The rotation-orbit tidal coupling tends to make elliptical orbits circular.

4. As we discuss in the following section, there is a limiting radius for the orbit of one body around another, inside of which the tidal forces are so large that no large solid objects held together solely by gravitational forces can exist. This radius is called the *Roche Limit.* Solid objects put into orbit inside the Roche Limit may be torn apart by tidal forces and, conversely, solid objects cannot grow by accreting gravitationally into larger objects if they orbit inside the Roche Limit.

11.3 The Roche Limit

How close can one body come to another before it is torn apart by tidal forces? If the body is held together *only by gravitational forces,* this answer is given by the *Roche limit:*



$$r_R = 2.4 \left(\frac{\rho_0}{\rho_*} \right)^{1/3} R$$

r_R = Roche Limiting Radius
ρ_0 = Density of massive body
ρ_* = Density of second object
R = Radius of massive body

Approximation: If $\rho_* = \rho_0$ then $r_R \sim 2.4\,R$

If the densities of the two bodies are equivalent the Roche Limit takes a simple form. In that case, it is just 2.4 times the radius of the massive object. One very often uses this simplified form of the Roche Limit equation to estimate the Roche limiting radius.

Example: Roche Limit and the Rings of Saturn

A famous consequence of the Roche limit is the rings of Saturn. The radius of Saturn is about 6×10^4 km. Assuming that $\rho = \rho_*$, the Roche limit radius is

$$r_\mathrm{R} \simeq 2.4 R_\mathrm{Saturn} \simeq (2.4)(6 \times 10^4) \simeq 140,000 \text{ km}.$$

Much of the ring system lies within this radius (only the G and E rings are at larger radius). Because the rings lie inside the Roche limit for Saturn, they cannot be solid objects held together only by gravity and must be composed of many small particles. Conversely, the tidal forces associated with the rings being inside the Roche limit keep the ring particles from condensing to form a moon.

Solid objects can exist inside the Roche limit (spacecraft or ourselves), but they must be held together by forces other than gravity. For spacecraft or our bodies, chemical forces between the atoms and molecules are much stronger than gravity.

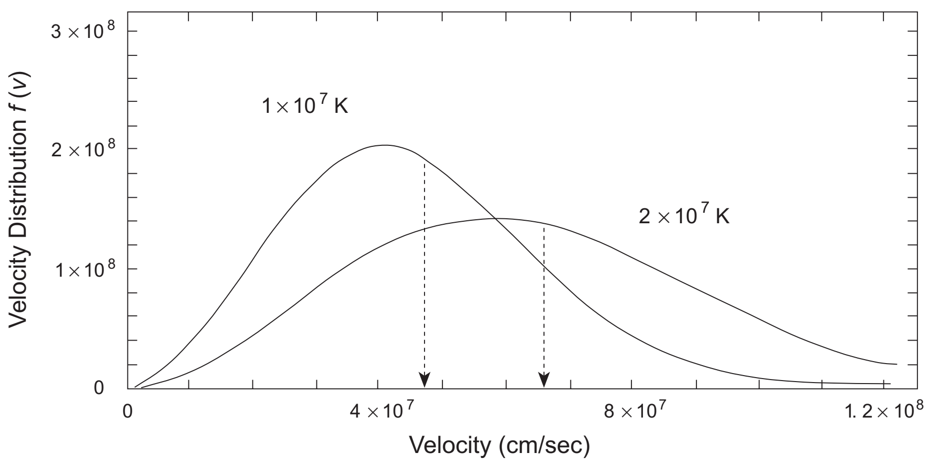
11.4 Retention of Planetary Atmospheres

Four basic factors governing whether a planet retains an atmosphere:

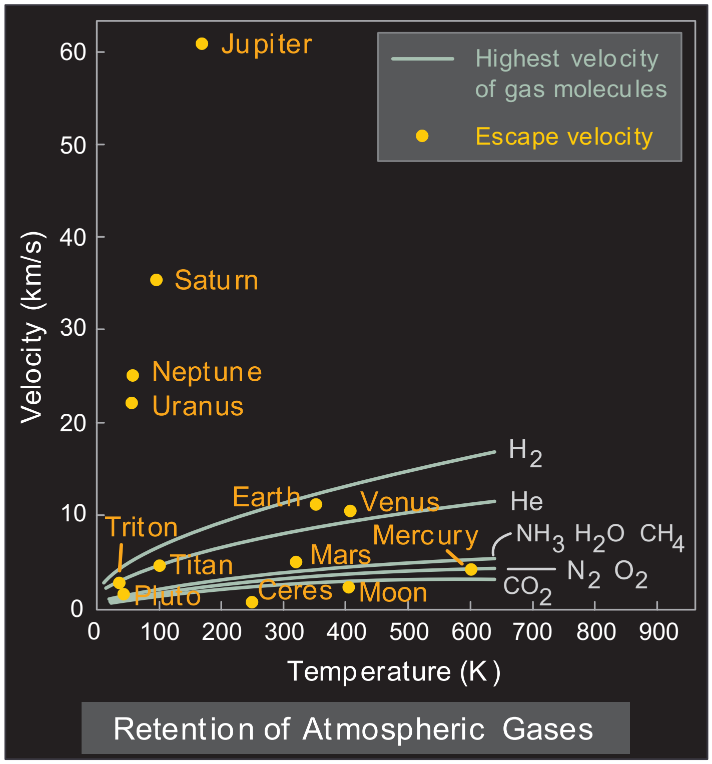
1. The mass of the planet
2. The mass of the gas molecules in the atmosphere
3. The temperature of the surface and atmosphere
4. The strength of the magnetic field.

The first factor governs how strongly the planet attracts the gas in its atmosphere, the second and third govern how rapidly the molecules move in the atmosphere, and the fourth determines whether there is a magnetosphere to shield the atmosphere from the solar wind.

Maxwell velocity distribution:



Show Virtual Astronomy Laboratory gas retention animation.
http://val.lightconeinteractive.com/vlabs/labs/vlab7/gasVelocity.html (val/vlabsdev)


Jupiter
Highest velocity of gas molecules
Escape velocity
Saturn
Neptune
Uranus
Velocity (km/s)
Earth
Venus
Triton
Mercury
Titan
Mars
Pluto
Ceres
Moon
H_2
He
NH_3 H_2O CH_4
N_2 O_2
CO_2
Temperature (K)
Retention of Atmospheric Gases